Self-Fulfilling Debt Crises, Revisited: 
The Art of the Desperate Deal*

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Abstract

Sovereign bonds occasionally exhibit sharp, large spikes in spreads which are only weakly correlated with movements in domestic output. We shed light on this and other empirical patterns using a quantitative model that incorporates familiar features, such as non-contingent bonds, endowment fluctuations, and shifts in creditor “beliefs” regarding the actions of other creditors. Different from the existing literature, our equilibrium includes self-fulfilling crises that involve the sovereign issuing small amounts of debt at strictly positive – but unusually low– bond prices. We refer to these auctions as “desperate deals,” and discuss counterparts in the data. The desperate deals involve arms-length transactions at competitive prices, and therefore must satisfy the same equilibrium requirements as non-crisis auctions. We support the prices by allowing the government to randomize when indifferent to default or repayment, generating volatility in spreads while matching the relatively rare frequency of default observed in the data. Contrasting this environment with the traditional Cole and Kehoe (2000) equilibrium, we show that the ability to auction at fire-sale prices during a crisis is crucial for generating volatility in spreads while also matching other key moments from bond price data.

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1 Introduction

In this paper we explore a novel class of self-fulfilling sovereign debt crisis equilibria. We build on the canonical Cole and Kehoe (2000) framework in which a coordination failure can lead to a “failed auction” and subsequent default. We extend this to incorporate self-fulfilling equilibria in which the sovereign auctions bonds at fire-sale – but strictly positive – prices. We motivate such “desperate deals” with experiences of emerging markets and recent European crisis countries, in which spreads are high and volatile but default remains relatively rare. The canonical Cole-Kehoe equilibria have difficulty explaining such episodes given the stark assumption that a crisis results in default with probability one. We explore quantitatively the differences between our framework and the canonical model, and show that including fire-sale auctions as part of the equilibrium path is crucial for understanding the high volatility of spreads.

The framework we explore builds on the standard Eaton and Gersovitz (1981) model and the recent quantitative versions beginning with Aguiar and Gopinath (2006) and Arellano (2008). In particular, the government of a small open economy faces endowment risk and issues non-contingent (but defaultable) bonds to a pool of competitive foreign investors. The creditors involved in sovereign lending are risk-averse with finite wealth, and hence the sovereign pays a risk premium. As in Cole and Kehoe (2000), our timing convention allows the sovereign to default in the same period as a successful auction. Cole and Kehoe used this timing to support an equilibrium price of zero for any amount of bonds sold at auction, which in turn is supported by immediate default due to the inability to rollover maturing bonds. Cole and Kehoe considered an equilibrium selection in which bonds are auctioned at a positive prices in non-crisis periods, but conditional on the realization of a sunspot, creditors coordinate on the zero-price equilibria, triggering default.

The idea that some factor other than domestic fundamentals, such as creditor beliefs about the equilibrium behavior of other lenders, is compelling. We motivate this feature by documenting a number of facts regarding emerging market and European bonds. First, there are two main traditions in the self-fulfilling debt crisis literature, associated with Calvo (1988) and Cole and Kehoe (2000). Loosely speaking, the former tradition focuses on multiple mappings satisfying equilibrium conditions between bond prices today and spending/default decisions in the future. See Lorenzoni and Werning (2013) and Ayres, Navarro, Nicolini, and Teles (2015) for recent papers in the Calvo tradition. The Cole and Kehoe (2000) model features multiple pairs of prices and contemporaneous default decisions that satisfy equilibrium conditions, with multiplicity reminiscent of a bank run. Recent papers in this tradition include Conesa and Kehoe (2011) and Aguiar, Amador, Farhi, and Gopinath (2015).
as is well known, emerging market spreads over benchmark risk-free bonds are volatile. Second, while large spikes in spreads are correlated with declines in output, the correlation is relatively weak. In fact, a sizable proportion of such spikes occur when growth is positive and in line with historical means. Third, we show that the same holds in the shorter sample of European crisis countries (Portugal, Ireland, Italy, Spain, and Greece). While the literature has shown some of the variation in spreads can be explained by shifts in measures of global risk premia, there remains a large and time-varying unexplained residual component. One possible interpretation of this residual source of risk is shifts in creditors beliefs about the behavior of other creditors.

As mentioned above, the failed auctions of the standard Cole-Kehoe model shed light on how creditor beliefs can play a role in generating defaults, and how this prospect affects government policy ex ante. However, in practice, sovereigns in crisis frequently escape default by issuing a minimal amount of bonds at low prices. As a motivating example, consider the case of Portugal. Yields on Portugal’s bonds increased in 2010. By the start of 2011, Portugal was in distress and having difficulty rolling over its maturing bonds. In January of 2011, it issued one billion euros in a “private placement” that was reportedly purchased by China. This was not sufficient to stem the crisis, and in May of that year Portugal began to draw on emergency funding from the EU. In late 2012, the prospect of bonds maturing in 2013 loomed. In anticipation, the Portuguese debt agency re-purchased bonds maturing in September 2013 while issuing bonds maturing in 2015. This swap was accomplished not through default, negotiation, and restructuring, but rather was implemented via a dual auction. The OECD Sovereign Borrowing Outlook 2013 referred to this type of transaction as “market-friendly solutions to resume market access and to ease near-term redemption pressures.” A benefit of the operation was to avoid the risk of a failed auction in 2013 when the original bonds matured. As it turned out, Portugal did successfully auction bonds in 2013, but did so without the threat of a rollover crisis due to the maturity swap.

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2 We are grateful for conversations with Pedro Teles regarding Portugal’s debt management during the crisis.
5 The Portuguese debt agency annual report for 2012 (http://www.igcp.pt/gca/?id=108) notes that “the management of the debt portfolio takes into account the refinancing profile of (IGCP) the debt, so as to avoid an excessive concentration of redemptions...” Its 2013 report states that its various operations “enabled the IGCP to accumulate levels of liquidity” which it used in part to reduce additional future commitments.
This narrative gives a sense in which a debt crisis involves a rich menu of possibilities, even in the absence of outright default and re-negotiation. We capture some of this richness in a tractable manner by incorporating “desperate deals” as part of the equilibrium outcome during a “coordination failure.” In particular, we follow Cole and Kehoe and introduce a sunspot that coordinates creditor beliefs between a relatively high equilibrium price schedule and a crisis price schedule. However, rather than the latter involving zero prices and immediate default, we consider an equilibrium price schedule which makes the government indifferent to default or repayment immediately after the auction. In our quantitative model, such prices typically imply spreads roughly 500 basis points higher than non-crisis periods, which is inline with many real-world episodes. This price schedule is rationalized by allowing the government to play a mixed-strategy over post-auction default, with the probability of default consistent with the original price schedule. As the government is indifferent, randomization is an acceptable best response to the equilibrium price schedule. In this sense, our approach corresponds to a worldview that debt crises push a sovereign to the brink of default, but whether default is actually realized is a random outcome that is independent of fundamentals, and, from the creditors’ perspective, a matter of luck.

A few features of this approach are worthy of note. The equilibrium price schedule and the government’s mixed-strategy response are part of a competitive equilibrium. While bargaining and re-negotiation are important aspects of sovereign default, many emerging markets and all European crisis countries other than Greece managed their crises without resorting to outright default. The auctions we consider are arms-length transactions involving competitive prices. Moreover, as the prices are competitive, they are not bailouts. While bailouts are a feature of many crises, the economics of official assistance are relatively well understood. Our focus is on the less familiar market transactions that occur during crisis periods.

Although the desperate deals do not involve bargaining or transfers, they do benefit legacy bondholders (compared to default), and deliver the default-value to the government without the associated deadweight costs of default. In this sense, conditional on the occurrence of a crisis, the deals raise the efficiency of bond markets. Given the competitive nature of the bond market, the sovereign reaps this gain ex ante through better prices. We show that this has important implications for welfare as well as the willingness of the government to borrow despite the prospect of crises. An important analytical insight of the Cole and Kehoe (2000) approach.

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[^6]: See Benjamin and Wright (2008), for example.
model is that potential of a crisis, and the associated ex ante equilibrium price schedule, induces the government to delever in order to avoid being vulnerable to a self-fulfilling run. Replacing failed auctions with desperate deals mitigates this tendency.

We calibrate the model to Mexico and quantitatively contrast our benchmark model with desperate deals to the canonical Cole-Kehoe framework in which crises generate certain default. With desperate deals, we match key bond market regularities, including the average and standard deviation of bond spreads, average debt-to-income ratios, and a default frequency of twice every one hundred years, the latter being consistent with broad historical samples. In the Cole-Kehoe version of the model, the standard deviation of bond spreads is a factor of twenty-five times too small. While that model generates frequent enough defaults, the sovereign never borrows into high spreads. In our benchmark model, the government is more willing to accumulate debt, and, more importantly, willing to issue bonds at fire-sale prices when faced with the crisis price schedule.

Using our benchmark model, we also contrast defaults due to a coordination failure versus “fundamental” defaults in which the government defaults despite the creditors coordinating on the better equilibrium price schedule. The latter have a distinct boom-bust pattern, in which default is preceded by abnormally high growth followed by a large negative growth realization. The high growth generates high bond prices, inducing the government to leverage up. The relatively high level of debt leaves the sovereign unwilling to repay when an abnormally low growth outcome is realized. This pattern shares something in common with the data, but our empirical work suggests booms followed by large recessions represent only a fraction of debt crises in practice. Moreover, fundamental defaults do not generate an ex ante spike in spreads, as the low growth realization is largely unanticipated given the preceding Markov process for the endowment.

The model’s defaults associated with coordination failures do not have an anticipatory boom and coincide with a relatively moderate contraction of endowment. Relatively high debt levels are also a necessary component of default, but in our benchmark these are frequently observed in the ergodic distribution due to the reasons discussed above. Given this vulnerability, a coordination failure generates a spike in spreads as the government issues bonds at “desperate deal” prices.

The benchmark simulations rationalize why large recessions may not yield large jumps in spreads, while smaller recessions can be associated with extremely adverse outcomes. In our motivational empirical work, we show this pattern is representative of a large sample of
sovereign debt crises. The model replicates this when three conditions coincide: (i) Relatively high debt-to-income; (ii) negative, but not abnormally so, growth; and (iii) creditors coordinating on the crisis equilibrium price schedule. Moreover, the simulated equilibria generate crises that, on average, are followed by immediate default only 15 percent of the time. Thus most large jumps in spreads are not associated with default outcomes, replicating the empirical fact that spikes in spreads are large and frequent in the data, but defaults are relatively rare.

The rest of the paper is organized as follows. Section 2 presents motivating facts from emerging markets and European crisis countries; Section 3 lays out the model; Section 4 discusses equilibrium selection and includes a detailed discussion of equilibrium behavior during a rollover crisis; Section 5 discusses calibration; Section 6 present the quantitative results; Section 7 contains a brief discussion of how including desperate deals compares with other approaches in the literature to mitigating the deadweight costs of default; and Section 8 concludes.

2 Motivating Facts

In this section we motivate our analysis by documenting the behavior of sovereign debt spreads in countries at risk of default. We begin with some motivating examples and then turn to a more systematic analysis of emerging markets. We conclude the section with a discussion of recent trends in Europe.

2.1 Motivating Examples

In Figure 1 we present data from Mexico and Italy, two economies frequently discussed in the context traditional emerging market crises and the recent European experience, respectively. The line in each figure is a measure of the sovereign debt spread; specifically, for Mexico this represents the EMBI spread over US dollar bonds and for Italy the 10-year spread over German bonds. The datasources are discussed in more detail below. The bars represent quarterly real growth; specifically, each bar depicts log quarterly real GDP minus the previous quarter’s value.

The case of Mexico, depicted in panel (a), shows the sharp spike in spreads in the fourth quarter of 1994, which coincides with the onset of the “Mexico Crisis” in which the country
Figure 1: Motivating Examples

(a) Mexico

(b) Italy

Note: Panel (a) plots the EMBI spread for Mexico (solid line, left axis) as well as quarter-to-quarter log real growth (bars, right axis). Panel (a) plots the 10-year Italian bond spread over Germany bonds and quarter-to-quarter real growth. See text for sources and details.

devaluated, suffered a large capital outflow and sharp recession. While Mexico did not default, it was assisted by an emergency loan package engineered by the US Treasury and the IMF. This episode motivated the model of (Cole and Kehoe, 1996, 2000). Spreads spiked again in 1998, during the emerging market crises triggered by the Russian crisis. In this case, Mexico’s domestic economy suffered only a mild slowdown. The global financial crisis of 2008 is also well represented, both in terms of spreads and output.

Panel (b) depicts Italy. Italy suffered a recession in 2008-09 during the financial crisis. However, spreads did not spike dramatically until 2012. While growth is never robust for Italy during the decade depicted, it is noteworthy that the more severe recession of 2008 is less reflected in spreads than the crisis of 2012. Moreover, the sharp decline in spreads coincides with ECB President’s famous “Whatever It Takes” speech of July 2012. One interpretation of this turnaround is that the ECB was stepping in as a lender of last resort to eliminate a self-fulfilling crisis. Of course, the counter interpretation is that the ECB was promising to bail out lenders on the equilibrium path, and spreads therefore are pricing in this transfer.

Several features of these episodes are noteworthy, which we explore more systematically in the next subsection. In particular, spikes in spreads occur regularly, but a deep recession
is neither a necessary nor sufficient condition of such crises. Moreover, and admittedly this point is more speculative, some fluctuations are consistent with shifts in “creditor beliefs” about the equilibrium behavior of other agents (including large players like the IMF or ECB).

2.2 Emerging Market Debt Crises

We start with a set of facts that guides our model of sovereign debt crises. Our sample spans the period 1993Q4 through 2014Q4, and includes data from 20 emerging markets: Argentina, Brazil, Bulgaria, Chile, Columbia, Hungary, India, Indonesia, Latvia, Lithuania, Malaysia, Mexico, Peru, Philippines, Poland, Romania, Russia, South Africa, Turkey, and Ukraine. For each of these economies, we have data on GDP measured in 2005 domestic prices (real GDP), GDP in US dollars measured in current prices and exchange rates (nominal GDP), gross external debt in US dollars (debt), and on market spreads on sovereign debt.

We pool data across this sample of emerging markets. Table 1 reports summary statistics of the pooled sample; Appendix Tables A1 and A2 report summary statistics for the sample for individual economies. Table 1 documents the high and volatile spreads that characterized emerging market sovereign bonds during this period. The average spread is 4.31%, and the standard deviations of the level and quarterly change in spreads are 6.76 and 2.29 percentage points, respectively. The 95th percentile of the quarterly change is 1.58.

We also report the annualized return for the EMBI+ index, which is 9.7% for the period 1993Q1–2014Q4. Specifically, this index is a value-weighted portfolio of actively traded, emerging country bonds constructed by JP Morgan. Purchasing this index in 1993Q1 and holding over the subsequent 21 year period generates an annualized return of nearly 10 percent. The EMBI+ index contains bonds of different maturities, but for reference the comparable return on 2-year and 5-year US Treasury bonds is 3.7 and 4.7%, respectively. This implies a realized premium of five to six percent over this period. Whether the realized return reflects the ex ante expected excess return depends on whether our sample accurately reflects the population distribution of default and repayment. While many of the countries in our sample have very high spreads, only two – Russia in 1998 and Argentina in 2001 –

8 Note that Russia defaulted in 1998 and Argentina in 2001, and while secondary market spreads continued to be recorded post default, these do not shed light on the cost of new borrowing as the governments were shut out of international bond markets until they reached a settlement with creditors. Similarly, the face value of debt is carried throughout the default period for these economies.

9 Data source for GDP and debt is Haver Analytics’ Emerge database. The source of the spread data is JP Morgan’s Emerging Market Bond Index (EMBI).
Table 1: Emerging Market Bonds: Summary Statistics

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean $r - r^*$</td>
<td>4.31</td>
</tr>
<tr>
<td>Std Dev $r - r^*$</td>
<td>6.76</td>
</tr>
<tr>
<td>Std Def $\Delta (r - r^*)$</td>
<td>2.29</td>
</tr>
<tr>
<td>95th Pctile $\Delta (r - r^*)$</td>
<td>1.58</td>
</tr>
<tr>
<td>Annualized Realized Return</td>
<td>9.7</td>
</tr>
<tr>
<td>Mean $B_t$ (percent)</td>
<td>46</td>
</tr>
<tr>
<td>Quarterly Corr($\Delta (r - r^*), \Delta y$)</td>
<td>-0.27</td>
</tr>
<tr>
<td>Quarterly Corr($r - r^*, %\Delta B$)</td>
<td>-0.19</td>
</tr>
<tr>
<td>Quarterly Corr($\Delta (r - r^*), %\Delta B$)</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Note: Summary statistics for the pooled EMBI+ sample. See text for details.

ended up defaulting on their external debt, while a third, Ukraine, defaulted on its internal debt in 1998. This reflects that defaults are relatively rare events; Tomz and Wright (2013) document over a larger sample that default occurs with an unconditional probability of roughly 2 percent.

The data from emerging markets can also shed light on debt dynamics during a crisis. Table 1 documents that the average external debt-to-annualized GDP ratio of 46%. This level is low relative to the public debt levels observed in developed economies. The fact that emerging markets generate high spreads at relatively low levels of debt-to-GDP reflects one aspect of the “debt intolerance” of these economies documented by Reinhart, Rogoff, and Savastano (2003).

Moreover, the data suggest that periods of above-average spreads are associated with reductions in the face value of gross external debt. The pooled correlation of spreads at time $t$ and the percentage change in debt between $t - 1$ and $t$ is $-0.19$. The correlation of the change in spread and debt is roughly zero. However, a large change in spread (that is, a crisis period) is associated with a subsequent decline in debt. In particular, regressing the percent change in debt between $t$ and $t + 1$ on the indicator for a crisis in period $t$ generates a coefficient of -1.6 and a t-stat of nearly 3. This relationship is robust to the inclusion of country fixed effects. This implies that a sharp spike in spreads is associated with a subsequent decline in the face value of debt. Of course these reduced form correlation mixes shifts in supply and demand. Nevertheless such statistics provide an overview of the relationship between spreads and the quantity of debt along the equilibrium path, which
is useful context for the simulations from our quantitative model. Note that the decline in
the level of outstanding debt is not always associated with a decline in the debt-to-GDP
ratio. From Figure 2 we know that many crises are associated with declines in GDP, which
frequently are larger than the percentage declines in debt.

An important question is whether debt crises are associated with domestic fundamentals.
Table 1 reports a generally negative correlation between the change in spreads and output
at quarterly frequencies. However, this statistic masks surprising heterogeneity across crisis
episodes. To explore this further, we consider the distribution of growth rates conditional
on whether the country is experiencing a “debt crisis.” We define a debt crisis to be a large
spike in spreads, regardless of whether the country defaults or not. From Table 1, we see
that the 95th percentile of the quarterly change in spread is 158 basis points. We define a
debt crisis to be a quarterly change in spreads that exceeds this threshold. By definition, 5
percent of the sample involves a debt crisis. However, this is not equally distributed across
countries. As shown in Appendix Table A1, five countries never have experience a crisis,
while nearly 20 percent of Argentina’s quarterly observations are above the crisis threshold.

Figure 2 depicts the density of GDP growth in crisis and non-crisis quarters. Specifically,
each quarter is assigned to a crisis or non-crisis category, depending on whether the change
in EMBI spread from last quarter exceeds the 95th percentile threshold. We then plot the
estimated kernel density of quarterly GDP growth for each subsample. In Panel (a), the
solid line depicts the kernel density of quarterly growth in real GDP in quarters defined
as non-crisis, while the dashed line depicts the corresponding histogram for crisis quarters.
Panel (b) redefines growth as the average growth over the preceding four quarters; that is,
if period $t$ is used to define crisis status (that is, $r_t - r_{t-1} > 1.58$), then growth is averaged
over quarters $t - 5$ and $t - 1$ (that is, $(\ln Y_{t-1} - \ln Y_{t-5})/4$). Panel (c) looks at subsequent
growth; that is, average growth between period $t$ and $t + 4$.

By comparing the two densities in Panel (a), one can see from the increase in the fre-
cquency of negative growth rates for countries experiencing crises relative to the overall distribution that there is a higher tendency for negative growth rates to be associated with crises. However, what is striking is the extent to which countries experiencing positive growth also experienced crises. Overall, the graph indicates some association between negative growth and debt crises, but the association is not strong. Specifically, while mean contemporaneous (quarterly) growth during a crisis is -1.2, as opposed to 1.0 during non-crisis quarters, nearly half (46 percent) of the crisis periods have strictly positive growth.
Figure 2: Sovereign Debt Crises and Growth

(a) Contemporaneous Growth

(b) Preceding-Year Average Growth

(c) Subsequent-Year Average Growth

Note: Each panel overlays two histograms (kernel densities) of GDP growth. The sample consists of 23 emerging markets between 1993Q4 and 2014Q4 (see text). The histogram labelled “No Crisis” refers to periods in which the quarterly change in sovereign bond spreads is less than 158 basis points. The histogram labelled “Crisis” refers to periods in which the change in spread is greater than or equal to 158 basis points. This threshold is chosen so that 5 percent of the periods are defined as “Crisis.” In Panel (a), growth is defined as the quarterly change in log GDP contemporaneous with the quarterly change in spreads used to define a crisis. Specifically, if a crisis occurs in quarter $t$ due to $r_t - r_{t-1} > 185$, where $r_t$ is the spread over the risk-free rate observed in quarter $t$, then growth is $y_t - y_{t-1}$, where $y_t$ is log GDP in quarter $t$. In Panel (b), growth is averaged over the year preceding the quarter used to define a crisis; that is $(y_{t-4} - y_{t-5})/4$. In Panel (c), growth is averaged over the subsequent year; that is $(y_{t+4} - y_t)/4.$
Not only do contemporaneous fundamentals have a hard time accounting for debt crises, they do an even worse job of forecasting debt crises. Specifically, in Panel (b) we see that growth is often quite positive in the year prior to the jump in spreads. Here the association between growth and crises is substantially weaker than with contemporaneous growth rates.

The lack of contemporaneous and lagged association between growth debt crises leaves open the possibility that it is the anticipation of future bad fundamentals that lead to a rise in spreads and debt crises. If such news shocks played an important role, then one would expect to see high spreads forecast negative future fundamentals. In the third panel we illustrate the lack of a tight connection between news about fundamentals and spreads by again graphing growth rate histograms, but now the density is for growth rates in the year following a debt crises. The number of crises followed by positive growth is remarkable given that there are many reasons to believe high spreads should suppress economic activity (see, for example, Neumeyer and Perri [2005]).

This evidence suggests that while negative GDP growth is correlated with increases in spreads, the association is rather weak. This fact complements the finding of Tomz and Wright [2007] that a significant fraction (roughly 40%) of defaults occur when GDP is above trend. It also echoes the analysis of Uribe and Yue (2006), which uses a VAR methodology to decompose the variance of spreads in several emerging markets. They find that roughly 15 percent of the variance at business cycle frequencies is due to domestic fundamentals, while 60 percent of spread volatility is a residual shock orthogonal to domestic fundamentals and US interest rates. While shocks to output are a natural starting point for understanding sovereign debt crises, the data suggest there is much more to the story.

### 2.3 The European Crisis

The above facts concerned emerging markets, which have generated most of the post-war debt crises. However, the recent crisis in Europe has renewed interest in debt crises in more advanced economies. In Figure 3 we plot the time series of spreads for the key crisis countries. The figure depicts the familiar pattern of spreads near zero (relative to Germany) and then a sharp spike up after 2010 and then a quick decline, although the precise timing and magnitude differ across the crisis countries.

Albeit with a small sample, we can perform an analysis with the European countries that parallels our emerging market analysis. In particular, Figure 4 depicts the pooled
Figure 3: European Spreads
Figure 4: Distribution of European Spreads

(a) Levels

(b) Quarterly Change

Note: This figure depicts the kernel estimate of the distribution of the level of spreads (Panel (a)) and the quarterly change in spreads (Panel (b)). The sample consists of 10-year bond yields for Portugal, Ireland, Italy, Spain, and Greece, minus the 10-year yield on German bonds, between 1999Q2 and 2015Q1. For Greece, we begin the sample in 2001Q2 to coincide with entry into the euro. All five countries are pooled in computing the distribution.

distribution of the spreads and the quarterly change in spreads for the euro crisis economies between the start of the euro in 1999 and 2015Q1. The crisis is depicted in the long right tail of the spread, and the dispersed tails of the change in spreads. The 95th percentile of the quarterly change in spread is 1.36, which is not that different from the 1.58 for emerging markets. We therefore define a “crisis” period as a quarter in which the spread increases by more than 1.36. As with emerging markets, while overall five percent of the country-quarters are associated with a “crisis,” the distribution across countries is not uniform. Greece is in crisis 14 percent of the quarters, while Italy and Portugal are in crisis for only two quarters and Spain for none.

Figure ?? depicts contemporaneous quarterly growth during crisis and non-crisis quarters for this pooled sample. As with emerging markets, while crisis growth is on average depressed, there is significant overlap between the two densities. Although the overlap is not as significant as is the case for emerging markets, we present these results as suggestive that the euro crisis is not fundamentally different in this regard than the more common emerging market crises. However, the small sample cautions that this evidence is only suggestive.
Figure 5: European Crisis and Growth

Note: The figure overlays two histograms (kernel densities) of GDP growth. The sample consists of Portugal, Ireland, Italy, Spain, and Greece between 1999Q2 and 2015Q1. For Greece, we begin the sample in 2001Q2 to coincide with entry into the euro. The histogram labelled “No Crisis” refers to periods in which the quarterly change in sovereign bond spreads is less than 136 basis points. The histogram labelled “Crisis” refers to periods in which the change in spread is greater than or equal to 136 basis points. This threshold is chosen so that 5 percent of the pooled country-periods are defined as “Crisis.”
2.4 Taking Stock

The empirical facts documented above suggest sovereign debt crises are more than just the result of poor fundamentals, which is the natural starting point of the quickly growing literature on sovereign default. A successful quantitative model should address the following empirical regularities: (i) 1. Interest rate spikes much more common than defaults; 2. Crises are not tightly connected to poor fundamentals; 3. Spreads are high and highly volatile; 4. Rising spreads are associated with de-leveraging by the sovereign; and 5. Coordination failures may be a relevant factor in sovereign bond markets. In the next section we introduce a quantitative model to shed light on the mechanism underlying these patterns.

3 Environment

We consider a single-good, discrete-time environment, with time indexed by $t = 0, 1, \ldots$. The focus of the analysis is a small open economy that receives a stochastic endowment. The economy is small relative to the rest of the world in the sense that its endowment realizations and decisions do not affect the world risk-free interest rate. However, financial markets are segmented in the sense that the economy can borrow from a set of potential lenders with limited wealth. Consumption and saving decisions are made on behalf of the domestic economy by a sovereign government. In this section, we proceed by characterizing the domestic economy and the sovereign’s problem, then turn to the lenders’ problem, and conclude by defining an equilibrium in the sovereign debt market.

3.1 Endowment

The economy receives a stochastic endowment $Y_t > 0$ each period. The endowment process is characterized by:

$$Y_t = G_t e^{z_t},$$ (1)

where

$$\ln G_t \equiv \sum_{s=1}^{t} g_s,$$ (2)

is the cumulation of period growth rates $g_t$, and $z_t$ represents fluctuations around trend growth. We assume that $g_t$ and $z_t$ follow finite-state first-order Markov processes. The
relevant state vector for the current endowment and its probability distribution going forward is \((Y_t, g_t, z_t)\).

### 3.2 Financial Markets

The sovereign issues non-contingent bonds to a competitive pool of lenders (described below). Bonds pay a coupon every period up to and including the period of maturity, which, without loss of generality, we normalize to \(r^*\) per unit of face value, where \(r^*\) is the (constant) international risk-free rate. With this normalization, a risk-free bond will have an equilibrium price of one. For tractability, we consider a bond with random maturity, as in \[\text{Leland (1994)}\]10 In particular, each bond matures next period with a constant hazard rate \(\lambda \in [0, 1]\). We let the unit of a bond be infinitesimally small, and let maturity be \(iid\) across individual bonds, such that with probability one a fraction \(\lambda\) of any non-degenerate portfolio of bonds matures each period. The constant hazard of maturity implies that all bonds are symmetric before the realization of maturity at the start of the period, regardless of when they were purchased. Note as well the expected maturity of a bond is \(1/\lambda\) periods, and so \(\lambda = 0\) is a console and \(\lambda = 1\) is one-period debt. While we vary \(\lambda\) across quantitative exercises, within any specific environment there is only one maturity traded. With these conventions, a portfolio of sovereign bonds of measure \(x\) receives a payment (absent default) of \((r^* + \lambda)x\), and has a continuation face value of \((1 - \lambda)x\).

We denote the outstanding stock of debt at the start of period \(t\) by \(B_t\). We do not restrict the sign of \(B_t\), which allows the government to be either a net creditor \((B < 0)\) or debtor \((B > 0)\). Net issuances of new debt in period \(t\) is given by \(B_{t+1} - (1 - \lambda)B_t\), where \((1 - \lambda)\) is the fraction of debt that does not mature in the current period. If \(B_{t+1} < (1 - \lambda)B_t\), then the government is repurchasing its outstanding debt rather than issuing new debt. We denote the debt-to-income ratio by \(b_t \equiv \frac{B_t}{Y_t}\). To rule out Ponzi schemes, we place an upper bound on the debt-to-income ratio: \(b_t \leq \bar{b}, \forall t\).

#### 3.2.1 Timing

The timing of a period is depicted in Figure 6. Let \(s\) denote the aggregate state after the period’s realization of random variables but before the period’s consumption and debt-
issuance decisions have been made. Specifically, $s = (Y, g, z, b, \rho)$, where $(Y, g, z)$ characterize the current period endowment state; $b$ is the inherited stock of debt normalized by $Y$; and $\rho$ is a random variable that coordinates equilibrium beliefs and is discussed in Section 4.3. As $Y$ is an element of $s$, the normalization of the other variables is without loss of generality. Let $S$ denote the set of possible states $s$. Other than $b$, the elements of $s$ follow an exogenous Markov process.

![Figure 6: Timing within a Period](image)

After observing the period’s realized $s$, the government decides to auction $B' - (1 - \lambda)B$ units of debt, where $B'$ represents the face value of debt at the start of the next period. There is one auction per period. While this assumption is standard, it does allow the government to commit to the amount auctioned within a period. Let $b'$ denote next period’s face value of debt normalized by the current endowment: $b'_t \equiv \frac{B'_t}{Y_t}$. The evolution of debt-to-income is then:

$$b_{t+1} = b'_t \frac{Y_t}{Y_{t+1}} = b'_te^{-g_{t+1}-z_{t+1}+z_t}.$$  

We postpone the formal definition of equilibrium until Section 3.5, but to anticipate we shall consider equilibria in which endogenous variables are functions of the state $s \in S$. Let $q(s, b')$ denote the equilibrium price schedule. The government is large in its own debt market and internalizes the fact that it faces different prices depending on how much debt it auctions; hence in choosing, $b'$ the government internalizes the entire price schedule as a function of $b'$. It is useful to define $x(s, b')$ as the equilibrium amount raised at auction per

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11 For an exploration of an environment in which the government cannot commit to a single auction, see Lorenzoni and Werning (2013).

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endowment, if positive, given an amount auctioned $b'$ and a price schedule $q(s, b')$:

$$x(s, b') \equiv \max \{q(s, b')(b' - (1 - \lambda)b), 0\}.$$  \hspace{2cm} (3)

The proceeds from auction are held in escrow until the government makes a repayment decision. In particular, the government can use these funds to pay its outstanding liabilities, but cannot draw on them for consumption unless all such payments are made. In particular, given outstanding debt-to-income $b$, the government is contractually obligated to pay $\lambda b$ in principal and $r^* b$ in interest payments. These payments are financed through current endowment as well as the revenue raised by the auction of new debt. If the government makes its contracted payments, it consumes per unit endowment $c \equiv \frac{C}{Y} = 1 - (r^* + \lambda)b + q(s, b')(b' - (1 - \lambda)b)$, and continues on to the next period with the new debt state implied by $b'$. We capture the value of the government in the repayment state by $V^R(s, b')$, which we define in the next subsection.

If the government defaults, the amount in settlement $x(s, b')$ is disbursed to all claimants on the basis of the face value of their claims. In particular, there are holders of current liabilities, totaling $(r^* + \lambda)b$, as well as holders of future liabilities, with a face value $b'$. In the period of default, each unit of such claims receives a payout $R^D$:

$$R^D(s, b') = \frac{x(s, b')}{b' + (r^* + \lambda)b}.$$  \hspace{2cm} (4)

If $b' < (1 - \lambda)b$, then the government has repurchased bonds on net. In this case we assume that the original holders of the repurchased bonds receive their payment at the time of the auction and there are no funds left in escrow at the time of default. In this case, $R^D(s, b') = 0$. The assumption that bondholders receive payments (if any) in proportion to the face value of their claims reflects the \textit{pari passu} and acceleration clauses typically in sovereign bond contracts.

In addition to losing any auction revenue, if the government fails to make all contracted payments in the period it enters the “default state” at the start of the next period. While in default, the government has no access to foreign financial markets. Moreover, in the default state the government loses part of its endowment, which proxies for economic disruptions that are a consequence of default in practice. Let $\phi(s)$ denote the proportional loss of output, so that in the default state the government receives $(1 - \phi(s))Y$ when the non-default state endowment is $Y$. While in default, the government has a constant hazard $\xi \in [0, 1]$ of exiting
the default state. Having exited default, the government no longer suffers an endowment penalty and regains access to foreign financial markets. The debt outstanding at the time of default is forgiven, implying that creditors are paid only the amount disbursed at settlement. We denote the expected value conditional on default by $V_D(s)$.

The timing places the auction before the default decision. An important implication of this convention is that holders of newly issued bonds are not fully compensated if the government defaults immediately after the auction. In this regard, our timing deviates from that of Eaton and Gersovitz (1981), which has become standard in the quantitative sovereign debt literature. In the standard timing, the bond auction occurs after that period’s default decision has been made. Thus newly auctioned bonds do not face within-period default risk. Our timing expands the set of equilibria relative to the Eaton-Gersovitz timing, and in particular allows a tractable way of introducing self-fulfilling debt crises. We discuss such equilibria in detail below, and in particular discuss the separate roles of the initial auction and the “Settlement” process by which auction revenue is dispersed and existing liabilities are (or are not) paid.

3.3 The Government’s Problem

The domestic economy is run by an infinitely lived sovereign government, which enjoys preferences over the sequence of aggregate consumption $\{C_t\}_{t=0}^{\infty}$ given by:

$$E\sum_{t=0}^{\infty} \beta^t u(C_t),$$

with $\beta \in (0, 1)$ and

$$u(C) = \frac{C^{1-\sigma}}{1 - \sigma},$$

with $\sigma \neq 1$. We assume that the sovereign has enough instruments to implement any feasible consumption sequence as a domestic competitive equilibrium, and therefore abstract from the problem of individual residents of the domestic economy. This does not mean that the government necessarily shares the preferences of its constituents, but rather that it is the

\footnote{The timing in Figure 6 is adapted from Aguiar and Amador (2014), which in turn is a modification of Cole and Kehoe (2000). The difference relative to Cole and Kehoe is that we do not allow the government to consume the proceeds of an auction if it defaults. See Auclert and Rogulic (2014) for a discussion of how the Eaton-Gersovitz timing in some standard environments has a unique Markov equilibrium, thus ruling out self-fulfilling crises.}
relevant decision maker viz a viz international financial markets.

To ensure that the government’s problem is well behaved, we require:

$$\max_g E_g \left\{ \beta e^{(1-\sigma)g'} \right\} < 1,$$

where the max is taken over elements of the Markov process for $g_t$.

Let $V(s)$ denote the start-of-period value for the government, conditional on the state $s$ and the equilibrium price schedule $q$ (which we suppress in the notation). Working backwards through a period, at the time of settlement the government has issued $B' - (1 - \lambda)B$ units of new debt at price $q(s,b')$ and owes $(r^* + \lambda)B$.

If the government honors its obligations at settlement, its payoff is:

$$V^R(s,b') = u(C) + \beta E \left[ V(s') \mid s, b' \right], \tag{5}$$

with

$$C = Y - (r^* + \lambda)B + q(s,b')(B'^* - (1 - \lambda)B$$
$$= Y \left[ 1 - (r^* + \lambda)b + q(s,b')(b'^* - (1 - \lambda)b \right].$$

Note that consumption is pinned down at settlement by the budget constraint; if the required consumption is negative, we define $V^R(s,b') = -\infty$, which is always dominated by default.

If the government defaults at settlement, its payoff is:

$$V^D(s) = u(C) + \beta(1 - \xi) E \left[ V^D(s') \mid s \right] + \beta \xi E \left[ V(s') \mid s, b' = 0 \right], \tag{6}$$

with

$$C = (1 - \phi(s))Y,$$

and where we recall that $\xi$ is the probability of exiting the default state. The output cost of default is governed by the function $\phi(s)$. Note that $s'$ includes $b'$ as an element, and $b' = 0$ while in the default state by definition. The amount of new debt implied by $b'$ is not relevant for the default payoff as the government does not receive the auction proceeds if it defaults at settlement.
The start-of-period value function is:

\[
V(s) = \max \left\{ \max_{b' \leq b} V^R(s, b'(s)), V^D(s) \right\}, \quad \forall s \in S.
\] (7)

Let \( B : S \to (-\infty, \bar{b}] \) denote the policy function for \( b' \) generated by the government’s problem. We denote the policy function for default at settlement conditional on \( b' \) by \( D(s, b') \). As we discuss in detail below (Section 4.2), we allow the government to randomize over default when indifferent; that is, when \( V^R(s, b') = V^D(s) \). Therefore, \( D : S \times (-\infty, \bar{b}] \to [0, 1] \) is the probability the government defaults at settlement, conditional on \((s, b')\). The policy function of consumption is implied by those for debt and default.

### 3.4 Lenders

We assume financial markets are segmented and only a subset of foreign agents participate in the sovereign debt market. This assumption allows us to introduce risk premia on sovereign bonds while treating the risk-free rate as parametric. For tractability, we assume that a set of lenders has access to the sovereign bond market for one period, and then exits, to be replaced by a new set of identical lenders. The short horizon of the specialist lenders is for tractability, avoiding the need to solve an infinite horizon portfolio problem and carry another endogenous state variable.

Specifically, each period a unit measure of identical lenders enter the sovereign debt market. Let \( W \) denote the aggregate wealth of the agents that can participate in the current period’s bond market. The entering “young” lenders allocate their wealth across sovereign bonds and a risk free asset that yields \( 1 + r^* \). As noted above, the risk-free rate is pinned down by the larger world financial market, and specialists in the sovereign bond market can freely borrow and lend at this rate.

Recalling the timing of Figure 6, “old” lenders enter a period with \( b \) units of debt (per endowment). A fraction \( \lambda \) of the representative portfolio matures, which is paid at settlement. We also assume that all coupon payments (on both maturing and non-maturing bonds) are paid at settlement. The remaining (ex-coupon) non-matured bonds, \((1 - \lambda)b\) are sold to “young” lenders at the time of auction. In particular, new lenders purchase from the legacy lenders the stock of non-maturing bonds plus any new bonds the government auctions at the same time. At the end of the auction, new lenders hold all non-maturing bonds.
With this timing, we can compute the return on bonds purchased in the current period by young lenders in state \( s \) when the government’s end-of-auction stock of debt is \( b' \). In particular, consider a young lender that purchases a unit-measure portfolio today, paying \( q(s, b') \) at auction. If the government defaults in the current period, the young lender receives \( R^D(s, b') \), where \( R^D \) is defined by [4]. As the lender is still young, it can invest this amount in risk-free bonds. If the government does not default this period, the young lender holds the sovereign bonds into the next period.

Next period, the lender is now “old.” It sells \( 1 - \lambda \) at auction, and receives \( q(s', b'') \), where \( b'' \) reflects next period’s debt issuance decisions. In equilibrium, this will be \( b'' = B(s') \). The lender receives \( q(s', b'')(1 - \lambda) \) for these bonds regardless of the government’s subsequent default decision. If the government does not default, the lender receives \( r^* + \lambda \) at settlement. Otherwise, it receives \( R^D(s', b'')(r^* + \lambda) \) at settlement.

Let \( \delta \) and \( \delta' \) denote indicator functions that take the value of one if the government defaults in the current or next period, respectively, and zero otherwise. The preceding implies that the realized return on a sovereign bond \( R \) purchased at price \( q(s, b') \) is given by:

\[
R = \frac{1}{q(s, b')} \left[ (1 - \lambda)q(s', b'') + \delta R^D(s, b')(1 + r^*) + (1 - \delta)\delta'R^D(s', b'')(r^* + \lambda) + (1 - \delta)(1 - \delta')(r^* + \lambda) \right].
\]

The first term on the right is the sale of non-maturing bonds at next period’s auction; the second term is the payment at settlement in case of immediate default, invest at the risk-free rate; the third term is the payment at settlement next period in case of default, scaled by the claims on coupons and matured principal; and the final term is the payment of coupon and principal absent default in either period. Note that \( q(s', b'') \) will be zero if the government defaults this period, and we therefore do not need to multiply by \( 1 - \delta \). Similarly, while that price incorporates the government’s default policy next period, the sale takes place before the default decision is made. Hence, it is also not multiplied by \( 1 - \delta' \).

Lender’s have preferences over wealth when old, \( W_o \), given by:

\[
v(W_o) = \frac{W_o^{1-\gamma}}{1 - \gamma}.
\]

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The young lender’s problem is to allocate a fraction \( \mu \) of its wealth in sovereign bonds, and the remainder in risk-free bonds. Given the homogeneity of preferences, the optimal decision conditional on \( s \) and \( b' \) is defined by:

\[
\mu^*(s, b') = \arg\max_{\mu} \mathbb{E} \left[ v((1 - \mu)(1 + r^*) + \mu R) \Big| s, b' \right],
\]

(9)

where \( R \) is given by (8). In forming expectations over \( R \), the lender uses the equilibrium policy functions of the government:

\[
\delta = 1 \text{ with probability } \mathcal{D}(s, b'); \\
\delta' = 1 \text{ with probability } \mathcal{D}(s', b'') \text{ in state } s'; \text{and} \\
b'' = \mathcal{B}(s').
\]

The first-order condition for the lender’s problem is the usual condition:

\[
\mathbb{E} M(R - (1 + r^*)) = 0,
\]

where \( M = v'((1 - \mu^*)(1 + r^*) + \mu^* R) \) is the stochastic discount factor. If lenders are risk-neutral, then we have that \( \mathbb{E} R = 1 + r^* \), which is the usual case in the quantitative literature. When \( \gamma > 0 \), we will have a positive risk premium. In particular, \( q(s, b') \) will be such that lenders receive the appropriate compensation for the probability of default plus any additional risk premium required to bear such risk. Note that the stochastic discount factor depends on \( \mu^* \). In equilibrium, the market for bonds must clear. In particular,

\[
\mu^*(s, b')W = q(s, b')b'.
\]

(10)

As \( b' \) increases, lenders devote more of their wealth to sovereign bonds, and therefore prices must fall to generate the appropriate risk premium to clear the market.

### 3.5 Definition of Equilibrium

**Definition 1 (Equilibrium).** An equilibrium consists of a price schedule \( q : S \times (-\infty, \bar{b}] \to [0, 1]; \) government policy functions \( \mathcal{B} : S \to (-\infty, \bar{b}] \) and \( \mathcal{D} : S \times (\infty, \bar{b}] \to [0, 1]; \) and a lender portfolio policy function \( \mu^* : S \times (-\infty, \bar{b}] \to \mathbb{R}; \) such that: (i) \( \mathcal{B} \) and \( \mathcal{D} \) solve the
government’s problem from Section 3.3, conditional on \( q \) and \( \mu^* \); (ii) \( \mu^* \) solves the representative lender’s problem\footnote{9}; conditional on \( q \) and the government’s policy functions; and (iii) market clearing: equation\footnote{10} holds for all \( s \in S \) and \( b' \in (-\infty, \bar{b}] \).

Note that the market clearing condition requires that the price schedule clears the market at all potential \( b' \), even for debt position that occur off equilibrium. This is a perfection requirement which ensures that if the government were to deviate from \( \mathcal{B} \) and issue a sub-optimal amount of debt, these bonds would be priced in a manner consistent with equilibrium behavior going forward.

4 Multiplicity

A primary target of the analysis is to explore quantitatively the role of self-fulfilling debt crises. We do so by exploiting the multiplicity of equilibria. We first discuss the nature of the multiplicity in our environment. We then turn to the ability of the government to issue bonds at very low prices out of desperation. Finally, we discuss equilibrium selection and how prices other than zero can be supported during a rollover crisis.

4.1 Rollover Crises

The multiplicity we focus on is static and involves the current price schedule offered to the sovereign. With an adverse price schedule, the sovereign can be induced to default today while with a generous price schedule it would not default for the same debt and output state variables. Moreover, the current and future default behavior of the sovereign in these two circumstances is consistent with the lenders optimally choosing to lend the prescribed amount at the prescribed price. We refer to a default with such an adverse price schedule as a rollover crisis. We refer to the case when the lender chooses to default with the generous schedule as a solvency default.

To see how this can occur in equilibrium, consider a state \( s \) such that \( q(s, b') = 0 \) for any \( b' \geq (1 - \lambda)b \). That is, the sovereign cannot raise money from the bond market in the current period. In the case that \( b' = (1 - \lambda)b \), which implies that the government did not issue new bonds when faced with a price of zero, then:

\[
V^R(s, (1 - \lambda)b) = u(Y[1 - (r^* + \lambda)b]) + \beta \mathbb{E}[V(s')|s, b' = (1 - \lambda)b].
\]
If $V^R(s, (1-\lambda)b) \leq V^D(s)$ then the government will find it weakly optimal to default at settlement. Moreover, in the equilibrium we consider, $V^R(s, b')$ is decreasing in its second argument, and so default is weakly optimal for any $b' > (1-\lambda)b$ when $q(s, b') = 0$.

On the other hand, for a given stock of debt and endowment, there may be a positive price schedule that can also be supported in equilibrium. That is, consider a pair $(\tilde{q}, \tilde{b})$ such that

$$u \left( Y \left[ 1 - (r^* + \lambda)b + \tilde{q}(\tilde{b} - (1-\lambda)b) \right] \right) + \beta \mathbb{E} \left[ V(s')|s, b' = \tilde{b} \right] > V^D(s).$$

This suggests that there may be an alternative price schedule at the same fundamentals such that the government does not default. That is, if it can raise revenue at auction to pay off maturing debt, it may not wish to default at settlement. Of course $\tilde{q}$ must be consistent with future default decisions to survive as an equilibrium, but this logic is how the model supports multiple equilibria. It also the same mechanism that underlies Cole and Kehoe (2000).

Note that whether a price of zero can be supported in equilibrium depends on income and inherited debt. It will be useful to define the “Crisis Zone” as those states in which a default is optimal when the government is faced with a zero price for new issuances:

$$\mathcal{C} \equiv \{ s \in S | u \left( Y \left[ 1 - (r^* + \lambda)b \right] \right) + \beta \mathbb{E} \left[ V(s')|s, b' = (1-\lambda)b \right] \leq V^D(s) \}. \quad (11)$$

Note that the left-hand side of the expression inside the braces is strictly decreasing in $b$. Moreover, in the equilibrium we consider, the left-hand side is increasing in $Y$ (that is, $g$ and $z$). Thus, the Crisis Zone consists of combinations of high debt and low endowment realizations. This will be useful to keep in mind when we analyze equilibrium defaults and the government’s endogenous response to the possibility of rollover crises.

## 4.2 Desperate Deals

While the canonical Cole and Kehoe (2000) model and the subsequent literature typically assumes that a rollover crisis implies a zero price and immediate default, the assumption of a complete inability to issue bonds at any positive price is somewhat extreme. As noted in the Introduction, countries may be able to turn to core lenders, such as primary dealers or small sets of fund managers, and issue small amounts of bonds at low, but positive, prices. We
incorporate this possibility by allowing for bond fire-sales during a rollover crisis, so called “desperate deals.” To do so we build on the mixed strategy equilibria introduced in Aguiar and Amador (2014), which allowed for off-equilibrium debt buybacks during a rollover crisis. We extend this idea to allow the government to issue debt during a rollover crisis on the equilibrium path.

Suppose that state $s$ is associated with a rollover crisis, which as noted above is defined by a low price schedule $q(s, b')$. In particular, suppose that $s \in C$, so that a zero price is consistent with equilibrium. If this were the equilibrium price, the government’s equilibrium payoff would be $V^D(s)$. Note that there are other prices that yield the same payoff to the government, but may not involve default. In particular, consider a schedule $q^D(s, b')$ defined by:

$$q^D(s, b') = \left\{ \tilde{q} \mid u\left( Y \left[ 1 - (r^* + \lambda)b + \tilde{q}(b' - (1 - \lambda)b) \right] + \beta E[V(s')|s, b'] = V^D(s) \right\}. \quad (12)$$

That is, $q^D$ traces out prices for candidate $b'$ such that the government is indifferent to defaulting. Given $b'$, note that the expression on the right is strictly increasing in $\tilde{q}$ for $b' > (1 - \lambda)b$, and strictly decreasing otherwise. Therefore there is always a price which satisfies the inequality. For $s \in C$, this price is non-negative by the definition of $C$. Below, we discuss further restrictions that ensure $q^D$ can be supported as an equilibrium price schedule.

Rather than assuming a price of zero in a rollover crisis, we can extend the notion of a crisis and consider a price schedule that satisfies (12) for some range of $b' > (1 - \lambda)b$. This allows the government to issue bonds at a positive, but very low, price. Alternatively, we could include fire-sale buybacks by considering $b' < (1 - \lambda)b$. Note that as $q^D$ traces out the locus that satisfies the indifference condition (12), the government’s optimal bond issuance choice is indeterminate. As part of an equilibrium selection, we therefore must specify how the government selects from this set.

Aguiar and Amador (2014) were concerned with the government repurchasing its debt if secondary prices were zero during a rollover crisis. They ruled these out by allowing the government to randomize over default when indifferent, supporting positive prices that made the government indifferent between repurchasing and defaulting. To make this an off equilibrium phenomenon, Aguiar and Amador explored the equilibrium in which the government always chose zero buybacks and default with probability one over buying back debt and randomizing.
4.3 Equilibrium Beliefs

The preceding discussion reflects that the set of allocations that can be supported in equilibria is very large. Our goal is to tractably restrict the set of equilibria and explore their quantitative properties in the hopes of shedding light on the possible mechanisms underlying the motivating facts presented above.

Following Cole and Kehoe (2000), we introduce a sunspot random variable that coordinates equilibrium behavior. In particular, let $\rho$ index the beliefs of agents, recalling that we introduced $\rho$ above as an element in the state vector $s$. We assume $\rho$ follows a two-state Markov process. If $\rho = r_C$, creditors coordinate on the rollover crisis equilibrium conditional on $s \in \mathcal{C}$, with the possibility of desperate deals. If $\rho = r_V$, the country does not experience a crisis this period, but is vulnerable to a crisis in the future.

Specifically, in the vulnerable belief regime ($\rho = r_V$), we construct an equilibrium price schedule that mimics the Eaton-Gersovitz schedule. That is, when $\rho = r_V$, it is as if the government can commit to the default decision for that period before the auction. In the lender’s problem (9), this corresponds to setting $\delta = 0$ for the current period, and solving for a price schedule that correctly anticipates the government’s equilibrium policy functions next period, given today’s $s$ and bond issuance $b'$. Given this price schedule, the government considers the bond issuance $b^* = \arg\max_\rho V^R(s, b')$. If $V^R(s, b^*) \geq V^D(s)$, then the government does not default. Otherwise, it defaults with probability one and does not auction any bonds. As noted above, within a period, this is the typical equilibrium fixed point in the quantitative Eaton-Gersovitz models. Let $q^{EG}(s, b')$ be the resulting price schedule for $\rho = r_V$. As in the Eaton-Gersovitz framework, this schedule is forward looking in the sense it is independent of inherited debt $b$. We then make this price consistent with our timing by letting $q(s, b') = (1 - D(s, b'))q^{EG}(s, b')$, where $D(s, b')$ is the optimal default policy conditional on facing $q^{EG}(s, b')$ this period.

If $\rho = r_C$ and $s \in \mathcal{C}$, then we assume creditors coordinate on the rollover crisis price schedule, augmented with desperate deals, $q^D(s, b')$. As noted above, $q^D(s, b') \geq 0$ for $s \in \mathcal{C}$. We also must verify that $q^D(s, b') \leq q^{EG}(s, b')$. That is, fundamentals may be so severe that

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14 Recall that with our settlement assumption, the government is indifferent to auctioning bonds when it plans to default, as it does not keep the proceeds. This marks a difference with the original Cole-Kehoe timing, in which the government could take the auction proceeds and then default.

15 One important caveat is that there is no guarantee that the Eaton-Gersovitz equilibrium price schedule is unique with longer maturity bonds. See Aguiar and Amador (2016). We follow the literature in this regard and iterate from the risk-free price schedule until we find a fixed point.
to make the government indifferent to defaulting we require a price higher than the \( \rho = r_V \) price. In this case, the sovereign will default regardless of beliefs and there is no equilibrium price schedule which makes them indifferent, and we set \( q^D(s, b') = 0 \).

One key difference between the traditional rollover crisis price of zero and the desperate deals price schedule is that bonds are trading at positive (albeit possibly low) prices in the midst of the crisis. To support such prices as an equilibrium, we allow the government to randomize over default at settlement. As the prices are selected to ensure the government’s indifference, mixed strategies are incentive compatible for the government. After auctioning \( b' \) in these states, the government chooses the probability of default at settlement, \( \mathcal{D}(s, \bar{b}) \in [0, 1] \), such that the lenders are willing to purchase \( b' \) at auction. That is, the mixed strategy probability is chosen to satisfy the bond-market clearing condition (10).

Summarizing the above discussion, the equilibrium price schedule is given by:

\[
q(s, b') = \begin{cases} 
(1 - \mathcal{D}(s, b'))q^{EG}(s, b') & \text{if } \rho = r_V \text{ or } s \notin C \\
q^D(s, b') & \text{if } \rho = r_C \text{ and } s \in C \text{ and } q^D(s, b') \leq q^{EG}(s, b') \\
0 & \text{otherwise.}
\end{cases}
\]  

(13)

From the price, we can compute the implied spread. Specifically, we define the spread as the implicit yield of a risk-free bond paying \( r^* \) each period and maturing with probability \( \lambda \) that is purchased at a price \( q(s, b') \).\[16\]

\[
r(s, b') = \frac{r^*(1 - \lambda) + \lambda}{q(s, b')} - \lambda.
\]  

(14)

As noted above, we must specify the debt issuance policy during a rollover crisis, as by definition that government is indifferent over a range of \( b' \). We assume that the government issues an amount equal to maturing debt, keeping outstanding debt constant. That is, \( \mathcal{B}(s) = b \) for \( s \in C \).

### 4.4 Notable Features of Desperate Deals

There are a number of noteworthy features of these “desperate deals” that arise from the competitive nature of the auctions. First, they occur at equilibrium prices. This is not a

\[16\] That is, \( q(s, b') = \sum_{k=1}^{\infty}(1 + r(s, b'))^{-(1 - \lambda)}(1 - \lambda)^{k-1}[r^*(1 - \lambda) + \lambda] \).
bargaining outcome in which creditors or the government threaten to “walk away.” The lenders have no incentive to hold out from the auction, as at the margin they are indifferent to participating or not. Thus, while this results in a positive price for legacy lenders who are selling their bonds at the same time, this is not a partial default or haircut in the usual sense.

Second, while the government is indifferent to the amount issued in a crisis at the “indifference” price schedule, the legacy lenders are not. In particular, they would like the government to choose an amount that maximizes the price of non-maturing bonds. However, given these are arms length transactions, there is no market mechanism to induce the government to select the surplus maximizing policy. This is the natural counterpart of the competitive assumption that there is no way for the government to “select” the equilibrium price schedule that does not involve a rollover crisis. Finally, and related to the last point, the ability to issue bonds in a crisis reduces the deadweight loss associated with rollover crises. In particular, depending on the realization of the mixed strategy randomization, some rollover crises are not followed with immediate default. Given a certain probability of a rollover crisis next period, the ability to engage in desperate deals will therefore raise bond prices ex ante. As we shall see, this encourages the government to borrow more in equilibrium and makes the bond market a more efficient provider of inter-temporal smoothing and insurance.

5 Calibration

We now calibrate the model and explore its quantitative predictions. We use Mexico as our reference economy.

Technology

For the endowment process, we assume the growth rate process is governed by

\[ g_{t+1} = (1 - \rho_g)\bar{g} + \rho_g g_t + \varepsilon_{t+1} \]

where \( \varepsilon \sim N(0, \sigma_g^2) \). The transitory component of output \( z_t \) is assumed to be iid, orthogonal to \( \varepsilon_t \) and to have mean zero and variance \( \sigma_z^2 \). The implied growth rate of log output is

\[ y_{t+1} - y_t = g_{t+1} + z_{t+1} - z_t + \varepsilon_{t+1}. \]
We estimate this model using quarterly Mexican constant-price GDP for the period 1980Q1 through 2015Q1. The estimated parameter vector is reported in Table 2. The estimates suggest that the stochastic trend is the primary driver of GDP fluctuations for Mexico, consistent with Aguiar and Gopinath (2007).

With these parameters in hand, we discretize the process for \( g \) using Tauchen’s method using 50 grid points spanning \( \pm 3\sigma_g/\sqrt{1-\rho_g^2} \). The \( iid \) \( z \) shock is drawn from a continuous Normal distribution truncated at \( \pm 3\sigma_z \). When taking expectations, we numerically integrate over \( z \)'s continuous distribution by evaluating at 11 grid points.

Preferences

The coefficient of relative risk aversion for the sovereign and the creditors is set to 2. The government’s discount factor is set through the moment matching procedure described below.

Financial Markets

We set the risk-free interest rate at 1 percent quarterly (hence, 4 percent annually). The average maturity length is set to 8 quarters, that is \( \lambda = 1/8 \), which implies a Macaulay duration of 6.4 quarters. This is shorter than the average maturity (or duration) observed in many emerging markets. However, maturity length is not constant over time and tends to shorten when the probability of a crisis is high (Broner, Lorenzoni, and Schmukler 2013, Arellano and Ramanarayanan, 2012). Moreover, much of a country’s short-term debt is issued domestically (whether in dollars or local currency), and thus is not reflected in the average maturity of external debt. For simplicity, our model only has external debt of constant maturity, raising the question of how to accurately capture a world in which maturity varies over time and the amount due (and to whom) in any given quarter is not uniform. Given our focus on crises, we set the average maturity length to a value that is relatively short.

The endowment while in default is \( Y_D(s) = (1-\phi(s))Y(s) \), where \( Y(s) \) is the endowment absent default. In particular, we set:

\[
Y_D(s) = \begin{cases} 
Y(s) & \text{if default occurs this period} \\
(1 - d)e^{z(s)}Y(s) & \text{otherwise}
\end{cases}
\]

where \( d \in (0, 1) \) is the adjustment to income due to being in default status. A few things are of note in this specification. One is that the cost of default is linear in output, as in Aguiar and
Gopinath (2006). In contrast, Arellano (2008) introduced a non-linear cost of default which made default disproportionately more costly in good endowment states and “forgiven” – at least in terms of output costs – in low endowment states. The Arellano specification amplifies the impact of endowment fluctuations in the decision to default while also making default a better insurance option in low-endowment states. This helps the model generate additional volatility of spreads and frequency of default, but does so by making endowment risk more important rather than less. The empirical facts outlined above and in complementary work like Tomz and Wright (2007) suggests that this pulls the model in the wrong direction relative to the data. The parameter \( d \) is set below by matching moments.

A second feature of \( \phi(s) \) is that no output is lost in the initial period of default. This is not a crucial assumption given the ability to adjust \( d \), but does simplify the exposition of how growth is correlated with the switch to default status. The timing reflects Figure 6 as default occurs at the end of a quarter, and hence output is not affected until the next quarter.

Finally, for computational convenience, we fix the transitory income shock to be a constant \( \bar{z} \) while in default. Thus \( Y(s) \) is adjusted by \( e^{\bar{z}-z(s)} \), where \( z(s) \) is the transitory shock associated with state \( s \). Given that \( d \) is a free parameter, the level of \( \bar{z} \) can be normalized to 0 without loss.

In addition to lost output, default also brings exclusion from financial markets. We set the re-entry probability after default to 0.125 quarterly; that is, the average duration of default is two years. This is in the range documented by Gelos, Sahay, and Sandleris (2011) for the 1990s, but lower than Tomz and Wright (2013)’s median of 6.5 years using a much longer sample. Again, this is not crucial given that we scale the value of default by choosing the parameter \( d \).

Simulated Matched Moments

We calibrate the remaining moments by simulating the model and matching targeted empirical moments. In particular, the remaining parameters are the probability of transiting from \( \rho = r_V \) to \( \rho = r_C \), and vice versa; the government’s discount factor; the wealth of the creditors, which we assume is a constant proportion of endowment \( w = \frac{W}{Y} \); and the proportion of output lost during default. We assume that the \( \Pr(\rho' = r_C | \rho = r_V) = \Pr(\rho' = r_C | \rho = r_V) \); that is, the probability of a crisis is independent of the current belief state. This leaves one transition probability and three parameters. We set these to match the average debt-to-GDP ratio, the average spread defined by (14), and the standard deviation.
of spreads, using Mexico as our empirical counterpart, as well as an average default rate of 2.0 percent per annum, which is in line with the estimates of Tomz and Wright (2013) using a broad sample of countries over a relatively long time period.

Specifically, for Mexico we match the average external debt to annual GDP (both in US dollars) for the period 2002Q1 through 2014Q3. The average over this period is 16.4 percent, which translates into a quarterly debt-to-income ratio of 65.6 percent. This measure of debt includes external debt by the government as well as banks. A longer time series exists for a narrower stock of debt issued by the federal government. This series suggests that debt levels were higher in the 1990s and have been falling in the 2000s and 2010s. Hence our measure of 65 percent may be an understatement. The average EMBI spread for Mexico over the entire period is 3.4 percent, with a standard deviation of 2.5 percent.

While moment matching involves simultaneously matching four moments by varying four parameters, we can provide a heuristic guide regarding which moments are particularly important for determining which parameter based on how the model behaves when we have varied parameters. In particular, the debt-to-GDP ratio is sensitive to the choice of the government’s discount factor $\beta$, which naturally governs the willingness of the government to borrow despite relatively high costs of doing so. Given a level of debt-to-GDP, the propensity to default is sensitive to the output loss parameter $d$. As we will discuss in detail below, the risk of a rollover crisis is important for generating the empirical volatility of spreads, which pins down the probability $\rho = r_C$. Finally, given the risk of default and spread volatility, the average spread reflects an average risk premium that is sensitive to lenders’ wealth.

These four targets, the model counterparts (under the column “Benchmark Model”), and the associated parameters are reported in Table 3. The model is simulated 1.5 million times, as default is a rare event. The quarterly spread defined by (14), with $b'$ evaluated at the equilibrium policy issuance, is converted into an annualized spread before computing simulated statistics. The model is able to hit all four targets precisely. To do so, the probability of a rollover crisis is fairly high; namely, fifty percent. As we shall see, this does not mean that a crisis occurs every other quarter on average. In particular, a crisis requires

17 In particular, $r^A(s, b') = (1 + r(s, b')^4 - 1$.
18 In the simulated model, the mean debt-to-income ratio and the spread is conditional on not being in the default state. More specifically, we compute the mean conditional on being out of the default state for at least 25 quarters. The reason we condition on being in good credit standing for a significant period of time is that the government exits default status with zero debt. Zero debt after default is not a realistic feature of the model and hence we focus on the ergodic distribution conditional on having sufficient time to rebuild debt.
that the debt is high enough and other fundamentals bad enough that a rollover crisis can be supported in equilibrium; that is, \( s \in C \). As debt is an endogenous state variable, the government has the ability to avoid a rollover crisis. This is will be a key element of the discussion to follow. Given the vulnerability to default in general and a rollover crisis in particular, we need a fairly low discount factor (0.82 quarterly) to ensure the government accumulates the target debt levels. Upon default, the government loses 7 percent of its endowment. The final parameter, creditor wealth to GDP, is a factor of 3.

Table 2: Parameters I: Set Prior to Simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endowments:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1 - \rho_g)\bar{g})</td>
<td>0.0034</td>
<td>Mexico GDP Data 1980Q1-2015Q1</td>
</tr>
<tr>
<td>(\rho_g)</td>
<td>0.445</td>
<td></td>
</tr>
<tr>
<td>(\sigma_g)</td>
<td>0.012</td>
<td></td>
</tr>
<tr>
<td>(\sigma_z)</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>Preferences:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sovereign CRRA ((\sigma))</td>
<td>2</td>
<td>Standard</td>
</tr>
<tr>
<td>Creditor CRRA ((\gamma))</td>
<td>2</td>
<td>Standard</td>
</tr>
<tr>
<td>Financial Markets:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quarterly Risk-free rate ((r^*))</td>
<td>0.01</td>
<td>Standard</td>
</tr>
<tr>
<td>Reciprocal of Avg. Maturity ((\lambda))</td>
<td>0.125</td>
<td>N/A</td>
</tr>
<tr>
<td>Default Re-entry Prob ((\xi))</td>
<td>0.125</td>
<td>Gelos et al (2011)</td>
</tr>
</tbody>
</table>

Note: Pre-set parameters for calibrated model.

The fact that \(u\) and \(v\) are homogenous functions and the budget set for the government’s problem is homogenous of degree one in \(Y\) implies that the level of endowment \(Y\) is not a relevant state variable for the equilibrium price schedule and associated policies, where recall that the policy functions for debt issuance and bond demand have been defined as ratios to current endowment \(Y\). Therefore we may solve for an equilibrium price schedule and the associated government’s and lender’s problems on a truncated (“detrended”) state space that

\[^{19}\text{See Alvare and Stokey (1998) for a formal treatment of dynamic programming with homogenous functions.}\]
Table 3: Parameters II: Simulated Method of Moments

<table>
<thead>
<tr>
<th>Target Moment</th>
<th>Benchmark</th>
<th>Model</th>
<th>No Deals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt-to-Income (Quarterly)</td>
<td>65.6%</td>
<td>65.6%</td>
<td>63.7%</td>
</tr>
<tr>
<td>Mean Annualized Spread ( r(s, b') )</td>
<td>3.4%</td>
<td>3.4%</td>
<td>3.1%</td>
</tr>
<tr>
<td>Quarterly Std Dev of Annualized Spread</td>
<td>2.5%</td>
<td>2.5%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Default Frequency (Annually)</td>
<td>2.0%</td>
<td>2.0%</td>
<td>1.9%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crisis Probability ( \Pr(\rho = r_C) )</td>
<td>50%</td>
</tr>
<tr>
<td>Discount factor ( (\beta) )</td>
<td>0.82</td>
</tr>
<tr>
<td>Default Cost ( (d) )</td>
<td>6.8%</td>
</tr>
<tr>
<td>Creditor Wealth Relative to ( Y(w) )</td>
<td>3.03</td>
</tr>
</tbody>
</table>

Note: The top panel reports the empirical moments and the model counterparts for our benchmark model and the alternative without “Desperate Deals.” The bottom panel are the values of the four parameters calibrated from matching the benchmark moments in the top panel to their empirical counterparts.

omits \( Y \). With these policies in hand, we can simulate the economy by drawing a sequence \( \{g_t, z_t\}_{t=0}^{\infty} \) and then iterating on the detrended policy functions to obtain equilibrium paths for debt-to-income as well as prices and default outcomes. Finally, the path of endowment \( Y_t \) can be constructed from \( \{g_t, z_t\}_{t=0}^{\infty} \), and the level of any endogenous variable can be obtained by scaling up its ratio with income.

6 Results

In discussing the quantitative results of the model, it is useful to contrast it with an environment in which the sovereign cannot issue bonds in a rollover crisis, as in much of the literature since Cole and Kehoe (2000). In Table 3 we add the moments from the alternative model in the column labelled “No Deals.” This model has the same parameters as the benchmark, but with a bond issuance policy of zero during crises (that is, \( b' = (1 - \lambda)b \)). Recall that the government is indifferent to the amount issued (or defaulting) when faced with the rollover crisis price schedule and thus zero issuance and default is also an equilibrium outcome of the model. In the “No Deals” equilibrium selection, conditional on a crisis the equilibrium price
is zero and the sovereign default’s probability is one.

The key equilibrium objects are the government’s debt issuance and default policy functions and the corresponding price schedule. In Figure 7 Panel (a) we plot the equilibrium price schedule as a function of debt issuances $b'$ in non-crisis periods; that is, $\rho = r_V$. Specifically, we plot $q^{EG}(s, b')$, which assumes the government does not default in the current period; in particular faced with $q^{EG}$, there exists a $b'$ such that $V_R(s, b') \geq V_D(s)$. We evaluate $s$ at the mean of $g$. As $q^{EG}$ is forward looking, it is independent of the inherited $b$ and the realization of the iid endowment shock $z$.

As is usual in these models, the price schedule is highly non-linear. The relevant region is in the neighborhood of the mean debt-to-income level of 65.6. Figure 8 depicts the ergodic distribution of $b$ in our simulated model, conditional on at least 25 quarters having passed since the most recent default. The figure indicates a fairly tight distribution around the calibrated mean of 65.5, a point we discuss below. In Panel (b) of Figure 7, we plot the price schedule over the tighter range relevant for the equilibrium debt distribution.

Figure 7 also depicts the “No Deals” alternative equilibrium price schedule. This schedule is shifted down relative to the benchmark, reflecting that forward looking lenders are pricing in the possibility of a rollover crisis and default; as the “No Deals” model generates a default during a crisis with probability one, ex ante prices are depressed. Correspondingly, in Figure 8 the distribution of debt is shifted to the left as well. Absent the desperate deals, the bond market is less efficient, and hence the government is unable to indulge its impatience to the same extent as in the benchmark model.

Figure 9 depicts the benchmark price schedule absent a crisis for three values of $g$; namely, the mean endowment realization and plus or minus three standard deviations of the unconditional distribution of $g$. The figure zooms into the relevant debt levels surrounding the mean $b$. A high realization of $g$ bodes well for future endowments, and thus future bond prices. Thus, the price schedule shifts up and out for high realizations of $g$. This reflects that in an incomplete markets environment, default is relatively attractive for low output realizations.

The nonlinearity of the price schedules in Figures 7 and 9 play an important role in equilibrium dynamics. Specifically, consider the revenue by auctioning $b' - (1 - \lambda)b > 0$ units of debt: $x(s, b') = q(s, b')(b' - (1 - \lambda)b)$. For the purposes of intuition, assume that $x$ is
Figure 7: Equilibrium Price Schedule: No Crisis

(a) Entire Debt Domain

(b) Relevant Debt Domain

Note: This figure depicts $q^{EG}(s,b')$ as a function of $b'$, evaluated at the mean $(z,g,b)$ and $\rho = r_V$; that is, the price schedule assuming no default in the current period. The solid line is the benchmark schedule and the dashed line is the “No Deals” alternative. Panel (a) depicts the entire debt domain, while panel (b) zooms into the domain that is relevant in the ergodic distribution.
differentiable in $b'$, and consider the revenue raised from an additional unit of debt:

$$\frac{\partial x(x,b')}{\partial b'} = q(s,b') + \frac{\partial q(s,b')}{\partial b'} (b' - (1 - \lambda)b).$$

The first term is the average price of bonds issued, while the second term captures that an additional unit of debt lowers the price of all bonds sold at auctioned. This latter effect reflects that all bonds are of equal seniority, and so the marginal bond has the same risk profile as the infra-marginal bonds. From Figure 7 we see that the second term is large and negative for debt issuances in the neighborhood of the mean debt-to-income level. As a consequence, despite the fact that $\beta << q(s,b')$ near the ergodic mean, the government does not wish to issue additional bonds in the nonlinear region of $q(s,b')$.

There are a number of important consequences of the preceding discussion. One is that the government does not venture far beyond the mean debt-to-income ratio in equilibrium, as shown in Figure 8. Second, debt issuances are pro-cyclical. From ??, we see that the nonlinear portion shifts to the right in response to a high $g$ realizations, as default is less likely going forward. This encourages additional borrowing, given the government’s impatience. Figure 10 depicts the policy function for the same three values of $g$ depicted in Figure 9. The high-$g$ policy lies above the mean-$g$ policy, which in turn lies above the low-$g$ policy. Moreover, near the 45 degree line, the policy functions are very flat. This indicates that the government levers up and down very quickly in this region in response to shocks to $g$. Finally, the nonlinear price schedule results in the government not borrowing enough to raise spreads substantially, as high spreads (low $q$) are associated with regions in which the price is highly elastic. This discourages borrowing and lowers the volatility of spreads. This generates a fairly low volatility of spreads absent the desperate deals, a point we discuss in detail below.
Figure 8: Ergodic Distribution of Debt

Note: This figure depicts the kernel density of debt-to-income for the benchmark model simulation (solid line) and the “No Deals” alternative. The distributions are conditional on no default within the last 25 quarters.

Figure 9: Benchmark Equilibrium Price Schedules: Response to $g$

Note: This figure depicts the benchmark $q^{EG}(s, b')$ as a function of $b'$ for different values of $g$. The top, dashed schedule is the highest $g$ in our discretization; specifically three standard deviations of the unconditional $g$ distribution above the mean. The lowest, solid line is three standard deviations below the mean, and the middle schedule corresponds to the mean $g$. The schedule is evaluated at $z = 0$, $\rho = r_V$, and $s = 65.6$, the ergodic mean.
Figure 10: Debt Issuance Policy Functions

(a) Full Domain

(b) Relevant Domain

Note: This figure depicts the bond issuance policy function $B$ as a function of $b$. The solid line is the benchmark policy, and the dashed line is the “No Deals” alternative. The schedules are evaluated at the mean values of $g$ and $z$ and for $\rho = r_V$. Panel (a) depicts the policy function over the entire debt domain, while Panel (b) focuses on the part of the domain relevant for the ergodic distribution.
Figure 11: Benchmark Debt Issuance: Shocks to $g$

Note: This figure depicts the benchmark model’s bond issuance policy function $B$ as a function of $b$ for various realizations of $g$, evaluated at $z = 0$ and $\rho = r_U$. 
The price conditional on a crisis is depicted in Figure 12. Specifically, the figure depicts $q^D$, which makes the sovereign indifferent to default. This is evaluated at the mean $g$, $z$, and $b$; note that in this case, the realization of $z$ and the level of $b$ matter, as the price schedule incorporates contemporaneous default.

The government’s policy function, the equilibrium price schedule, and the stochastic processes for endowment combine to generate a spread distribution. The ergodic distribution is depicted in Panel (a) of Figure 13. Most of the distribution is concentrated around the mean spread of 3.4%, with a long right tail during rollover crises. Panel (b) zooms in on the non-crisis part of the distribution by plotting the distribution conditional on $q(s,b') = q^{EG}(s,b')$; that is, no crisis. Absent a crisis, there is a fairly tight distribution of spreads, a feature we discuss below in detail. In Panel (b) we also plot the spread distribution of the “No Deals” alternative. Absent deals, the figure depicts the full distribution of spreads absent default, as crisis periods always generate defaults and a price of zero. The “No Deals” distribution is similar to the conditional distribution of the benchmark, indicating the importance of crisis deals in generating the volatility of the spread in the benchmark model.

Figure 14 considers spreads during crisis events in the benchmark model. Panel (a) plots the distribution conditional on a rollover crisis. We separate the events that result in repayment from those in which the randomization comes up default. As required by equilibrium, the crises with a higher spread are more likely to generate a subsequent default. Moreover, the distribution covers a range of spreads that encompass magnitudes observed during events like Mexico 1994 and Greece 2012. Panel (b) depicts the distribution of the randomization probabilities during crises. The mean of this distribution is 0.15; that is, conditional on a rollover crisis, the sovereign defaults 15% of the time.

In the benchmark, the sovereign is in a crisis quarter, that is $\rho = r_C$ and $s \in C$, only 2.6% of the time. This is despite the fact that the exogenous probability of $\rho = r_C$ is calibrated to be 50 percent. The rarity of rollover crises therefore reflects the fact that the government avoids the Crisis Zone. Exposing itself to a rollover crisis is costly ex ante due to the equilibrium price schedule. A rollover crisis therefore requires both a high level of debt, but also a relatively negative growth shock.

To see this, Figure 16 depicts the distribution of growth conditional on a rollover crisis and conditional on a non-crisis event. The crisis distribution is shifted to the left, indicating that self-fulfilling crises in our model involve a combination of a shift in beliefs and a negative
Note: Crisis bond price schedule, \( q^D(s, b') \), as a function of \( b' \) evaluated at \( g = -0.0212, z = -0.0008 \) and \( b = 0.652 \).
Figure 13: Ergodic Distribution of Annualized Spreads

(a) Unconditional

(b) Conditional on No Crisis

Note: This figure depicts the simulated distribution of spreads. Panel (a) depicts the distribution of spreads in the benchmark model including rollover crises. Panel (b) depicts the distribution of spreads conditional on no rollover crisis for the benchmark model (solid line) and the “No Deals” alternative (dashed line).

Figure 14: Crisis Spreads

(a) Annualized Spreads

(b) Randomization Probabilities $\mathcal{D}(s, b')$

Note: Panel (a) depicts the histogram of spreads in the benchmark model conditional on a rollover crisis. The unfilled bars denote episodes that did not result in a default, while the shaded bars depict the distribution conditional on a subsequent within-period default at settlement. Panel (b) depicts the simulated distribution of the government’s mixed-strategy probability of default, $\mathcal{D}$, during rollover crises.
output realization. The mean growth conditional on a crisis is xx, compared to xx for non-crisis quarters. Moreover, xx% of rollover crises are associated with negative growth. Figure 16 is comparable to the empirical counterparts depicted in Figures 2 and ???. In particular, while rollover crises are associated with relatively worse growth, there is a substantial overlap with growth during non-crisis periods.

6.1 Interest Rate Crises

To explore such episodes further, Table 4 explores the simulated economies conditional on an “interest rate crisis,” which, as in the data, we define as a change in spread in the upper 5th percentile of the ergodic distribution. For both the benchmark and the “No Deals” comparison, the 95th percentile is essentially the same; namely, 0.3%. However, recall that in the alternative model, there is no right tail to the distribution, while the benchmark has a long tail of roughly 2.5% of the quarters that occur during rollover crises. This contrast can be seen by the mean spread and mean change in spread, conditional on exceeding the
Table 4: Interest Rate Crises

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>No Deals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional Mean $r - r^*$</td>
<td>11.5</td>
<td>3.3</td>
</tr>
<tr>
<td>Conditional Mean $\Delta(r - r^*)$</td>
<td>8.3</td>
<td>0.4</td>
</tr>
<tr>
<td>Conditional Mean $\Delta y$</td>
<td>-1.3</td>
<td>-0.3</td>
</tr>
<tr>
<td>Conditional Fraction $\Delta y &lt; 0$</td>
<td>0.859</td>
<td>0.627</td>
</tr>
<tr>
<td>Conditional Mean $b$</td>
<td>66.4</td>
<td>63.8</td>
</tr>
<tr>
<td>Conditional Fraction $\rho = r_C$</td>
<td>0.497</td>
<td>NA</td>
</tr>
</tbody>
</table>

95th percentile threshold. In the benchmark the conditional mean spread is 11.5%, while the alternative is 3.3%. The latter is striking given that the ergodic mean for the alternative model is 3.1%. Similarly, the mean change in spread is 8.3% for the benchmark and only 0.4% for the alternative.

The table also indicates that negative growth is associated with negative crises, but slightly more so for the benchmark model. The correlation with growth reflects that fundamentals matter even during a self-fulfilling crisis. Similarly, the conditional mean of debt-to-income is higher than the unconditional mean, reflecting the sensitivity of spreads to debt levels. In particular, the alternative model’s conditional mean of 63.8 is only slightly above the unconditional mean of 63.7 reported in Table 3. Finally, for the benchmark model, roughly half of interest rate crises are associated with rollover crises and associated “desperate deals.” In the alternative, such crises always generate default and thus spreads are not defined.

6.2 Default Post-Mortems

We now turn to default episodes in the benchmark model. Table 5 reports the same statistics as in Table 4, but in this case the conditioning event is a default in the current quarter. For the benchmark, we can see that spreads spike during a default episode and growth is relatively low. Moreover, we see that 84.5% of defaults coincide with a rollover crisis. In the “No Deals” alternative this fraction is nearly 100%.

To obtain a better sense of the nature of rollover-crisis defaults, in Figure 17 we perform
an event study analysis in the benchmark model’s simulation. In particular, we normalize $t = 0$ as the quarter of default and then explore mean behavior in the preceding five quarters. The solid line depicts default events that occur with a rollover crisis ($\rho = \rho_V$), which we label “self-fulfilling” defaults. The dashed line depicts defaults that occur outside a rollover crisis ($\rho = \rho_V$). We label the latter defaults as “fundamental” defaults as the default would occur that period regardless of the realization of $\rho$; of course, the fact that future crises could occur play a role in the default decision today as these events are embedded in the value of repayment.

Panel (a) of 17 plots the mean growth leading up to a default event. For “fundamental” defaults we see a boom-bust pattern. Two quarters prior to default tends to generate high growth, which is then followed by a mediocre growth realization the period before default. The default itself coincides with a large negative growth realization. This pattern is the classic fundamental driven default; the high growth induces the government to borrow, and then if a large negative growth shock occurs while the economy is so highly leveraged the sovereign defaults. For self-fulfilling defaults ex ante growth is not particularly elevated, and default itself coincides with a mildly negative growth realization. The self-fulfilling defaults are thus associated with relatively milder recessions.

Panel (b) depicts the trajectory of debt before default. We see the increase in debt typical before a fundamental default, which reflects the boom period just discussed. A relatively high debt level is also necessary to sustain a self-fulfilling crisis in equilibrium, although the level is less than that associated with a fundamental default.

Finally, Panel (c) depicts spreads. For fundamental defaults, there is hardly any increase in spreads prior to the default. The fundamental defaults combine the shift up in the price schedule during the boom period and the sovereign’s best response of adding debt in response, keeping spreads largely unchanged. The default then occurs because an unusually large negative growth rate is realized after a relatively large positive growth rate; as low growth is relatively unlikely to follow high growth, spreads do not anticipate the fundamental default (other than the unconditional risk in all quarters). This contrasts with self-fulfilling crises, in which spreads spike in the quarter of the default as the government issues debt at very low prices. Creditors understand the risk of imminent default and charge accordingly.
Figure 17: Default Event Studies

(a) Growth

(b) Debt/Income

(c) Spreads
Table 5: Defaults

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>No Deals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional Mean $r - r$</td>
<td>18.4</td>
<td>NA</td>
</tr>
<tr>
<td>Conditional Mean $\Delta(r - r^*)$</td>
<td>13.9</td>
<td>NA</td>
</tr>
<tr>
<td>Conditional Mean $\Delta y$</td>
<td>-1.7</td>
<td>-2.2</td>
</tr>
<tr>
<td>Conditional Fraction $\Delta y &lt; 0$</td>
<td>0.973</td>
<td>0.999</td>
</tr>
<tr>
<td>Conditional Mean $b$</td>
<td>66.7</td>
<td>65.1</td>
</tr>
<tr>
<td>Conditional Fraction $\rho = r_C$</td>
<td>0.845</td>
<td>0.994</td>
</tr>
</tbody>
</table>

7 Discussion: The Value of Desperate Deals

We now discuss why desperate deals have a significant impact on how the bond market operates. Viewed in isolation, the ability to issue debt at the $q^D$ price schedule is immaterial to the sovereign; in both the benchmark and the canonical “No Deals” equilibrium, the government’s payoff is the default value. Similarly, as the deals occur at competitive prices, the new lenders are indifferent at the margin to buying bonds in periods of distress. However, the legacy lenders strictly prefer to a crisis equilibrium with a positive price for non-maturing bonds. In this sense, adding desperate deals generates a more efficient outcome conditional on a rollover crisis (and, of course, an even better outcome is generated by no crisis altogether).

Looking forward to the possibility of a positive secondary market price for their bonds, the legacy lenders are more willing to purchase bonds ex ante. This encourages the government to indulge its impatience and borrow, raising its non-crisis welfare. In particular, the sovereign is more likely to borrow into the crisis region, generating the extreme fluctuations in spreads observed in the data.

There are other ways to improve the efficiency of the bond market in models of this type. Looking at equilibria absent rollover crises is one approach, but this removes the high volatility at the same time. For example, Aguiar and Gopinath (2006) explore default in a model with a similar endowment process but no self-fulfilling crises and find a very stable spread.

Contrasting with this is the approach taken by Arellano (2008), who introduces nonlinear default costs. Specifically, defaults occurring during low endowment realizations are not
associated with an output loss, reducing the deadweight cost of default. Moreover, this makes non-contingent bonds better insurance, as default is partially “forgiven” when output is low (which is the typical default scenario). The sovereign is thus willing to borrow at fairly high spreads, generating empirical volatility of spreads. However, this requires a volatile output process (Arellano calibrates to Argentina). [Aguiar, Chatterjee, Cole, and Stangebye (in process)] calibrate an Arellano-type nonlinear default cost using Mexican data and find much of the volatility in spreads disappears. Moreover, as default costs are sensitive to output, this increases the importance of income in spread fluctuations, contrasting with the modest role discussed in Section 2.

Efficiency could also be enhanced by renegotiating under the threat of default. If bargaining is efficient, this eliminates the deadweight costs of default. [Yue 2010] explores such a model using an endowment calibration similar to [Aguiar and Gopinath 2006] and generates volatile spreads. Yue’s renegotiations happen immediately, and thus the sovereign is never punished for default on the equilibrium path. However, [Benjamin and Wright 2008] document that in practice it takes many years for defaults to be resolved. Thus introducing bargaining during default may not spare the sovereign the costs of default. However, in practice defaults are resolved through partial repayment, which compensates creditors. Our “Desperate Deals” scenario shares this aspect, although the partial payments occur through competitive secondary market trades rather than default resolution. This feature is compelling as many crisis episodes are not associated with default; in our sample, only Greece, Argentina, and Russia defaulted on their external bonds.

This discussion of making bond markets more efficient highlights a key aspect of the models. While adding a sunspot and self-fulfilling crises conceivable can generate a volatile bond market, the sovereign always has the option of avoiding the drama by not borrowing. This is what happens in our “No Deals” alternative. While the government defaults in response to a combination of high debt, low output, and a run on their bonds, the sovereign never ventures into the region of the state space in which spreads are particularly volatile. Impatience is not enough, as the nonlinear prices essential ration the feasibility of government debt. Viewed in this way, the volatility associated with real life markets must be relatively benign to support creditors willing to lend and government willing to borrow. In this study, the mitigating factor is the ability to issue bonds at fire-sale prices in crisis episodes.
8 Conclusion

In this paper we extended the nature of self-fulfilling crises to include bond issuances at fire-sale prices during a rollover crisis. This was motivated by the empirical behavior of spreads, and their relationship to other fundamentals, in emerging markets and European crisis countries. The addition of these “desperate deals” change the nature of equilibrium spreads in the quantitative model, particularly raising the volatility of spreads. Absent such deals, the volatility of spreads is an order of magnitude too small, despite the presence of self-fulfilling crises and defaults as frequent as in the benchmark. In the no-deals model, the sovereign either deleverages or defaults in response to adverse credit conditions. With deals, the government is willing to endure the high and volatile spreads due to crises as they are indifferent to repayment and default in such situations. However, creditors strictly prefer the positive prices of such deals, conditional on a crisis, and thus are willing to purchase bonds ex ante at more favorable prices for the issuer. This latter effect induces more borrowing on the part of the government as well as higher ex ante welfare, despite the volatility of spreads. The nature of desperate deals in the model, and the associated equilibrium behavior, provides a lens to interpret the interest rate crises used to motivate the analysis.
References


## Appendix Tables

### Table A1: Sovereign Spreads: Summary Statistics

<table>
<thead>
<tr>
<th>Country</th>
<th>Mean $r - r^*$</th>
<th>Std Dev $r - r^*$</th>
<th>Std Dev $\Delta(r - r^*)$</th>
<th>95th pct $\Delta(r - r^*)$</th>
<th>Frequency Crisis</th>
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Table A2: Sovereign Spreads: Summary Statistics

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