Planning Costs and Price Discrimination in Markets for Time-Specific Goods

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Abstract

When consumers have constant or non-existent planning costs firms can always achieve maximum profits with a single fixed price. However, when planning costs vary with consumers’ valuations a single fixed price is not always sufficient for profit maximization. In such cases it is optimal for the firm to provide an array of purchasing options. High valuation consumers will purchase with certainty at a high price while lower valuation consumers will opt for a lottery which gives some probability of being allowed to purchase at a discounted price and no opportunity to purchase otherwise. In general this type of mechanism may be difficult to implement, but in markets for time-specific goods any menu of lotteries can be realized by a descending price path with quantity restrictions at the discount prices.
1 Introduction

In some industries, such as the software and publishing industries, descending prices are an effective and well-known means of price discrimination. One common characteristic of these industries is that it is costly for high valuation consumers to wait for prices to drop before purchasing. For instance, having new software today is significantly more valuable than owning the right to begin using that software three months from now. This difference is instrumental in allowing firms to price discriminate via falling prices. High valuation consumers are willing to pay the high price now only because the added value of having the good now versus three months from now is greater than the discount they could have received by waiting.

However, this is not the case with time-specific goods such as airline tickets. A consumer should have no preference between purchasing an airline ticket today and purchasing a contract to receive the exact same ticket three months from now. Receiving the ticket early grants no additional value. For this reason, price discrimination through descending prices should be more difficult in markets for time-specific goods.

This conclusion is clearly manifested in the airline industry where prices usually rise rather than fall as the date of travel approaches. However, it is interesting to note that consistently declining prices can nevertheless be observed in some markets for time-specific goods. The most notable instance is likely the sale of Internet banner advertisements, but particular airline routes exhibit this tendency as well. One explanation for declining prices in these markets relates to demand uncertainty. If demand is uncertain and firms must make quantity decisions before demand is known, then firms will have an incentive to cut prices when demand is lower than anticipated in order to sell their excess capacity. When this explanation applies, one would expect the decline in price to be smooth and gradual so that the price is lowered just enough to sell off excess capacity. However, this is not always the case.

This paper attempts to explain why price paths characterized by one or more sudden price
drops, rather than smoothly declining prices, might be observed in markets for time-specific goods. I show that suddenly declining prices with quantity restrictions to induce rationing achieves optimal price discrimination when some consumers must plan and planning costs vary. This is true even when demand is certain. The following example illustrates this point.

Example 1 There are two types of consumers interested in purchasing a time-specific good that can only be used at time \( t = 1 \). Type A consumers value the good at 100 units but must pay a planning cost of 50 at time \( t = 0 \). Type B consumers value the good at 20 units but do not have any planning costs. Each group of consumers is of mass 1/2 and thus the total mass of consumers is 1. Marginal cost for the firm is zero. If the firm is restricted to posting a single fixed price it will sell the good at a price of 50. Only the type A consumers will buy which results in profits of 25. However, without this restriction the firm can do better. If the firm offers consumers the choice between purchasing with certainty at a price of 50 and having a 5/8 chance of buying the good at a price of 20 then type A consumers will all still buy with certainty at the price of 50 but now type B consumers will be able to buy some of the time as well. This results in profits for the firm of \( \frac{1}{2} \times 50 \) from the type A consumers and \( \frac{1}{2} \times \frac{5}{8} \times 20 \) from the type B consumers for total profits of 31.25. Consumer surplus is zero in both cases and thus total surplus and producer surplus both increase from 25 to 31.25.

Explicit implementation of the choice between purchasing with certainty at a high price and having a chance to purchase at a discount price may be both difficult and undesirable. However, the firm can achieve the effect of giving consumers these two options by varying the price over time: the firm starts by making an unlimited quantity available at a price of 50. At a random time \( t \in (0, 1) \) the firm then drops the price to 20 but restricts sales to mass 5/16. Consumers can either purchase immediately at the high price or wait for the price to drop before purchasing. By selecting the first option the consumer receives the good with certainty at a price of 50 while the second option gives the consumer a chance of being
allowed to purchase at a price of 20. However, since the demand at the low price is 1/2 while the supply is only 5/16 each consumer has only a 5/8 chance of being able to purchase at this price assuming that the rationing mechanism treats all consumers equally.

This example illustrates that optimal pricing cannot always be realized by a single fixed price. In such cases optimal pricing requires that consumers be allowed to choose between purchasing the good with certainty at a high price and selecting one of several lotteries. When a consumer selects one of these lotteries over purchasing with certainty at the high price, they receive with probability less than one the opportunity to purchase the good at a discounted price. When more than one lottery is available, a lower chance of being able to purchase the good corresponds to a more discounted price of purchase.

The intuition behind such a pricing mechanism is that it allows the firm to price discriminate between high valuation and low valuation consumers. Those with high planning costs will be prevented from purchasing at the discount price by the risk that they will be denied the opportunity to purchase the good and that their plans will be disrupted. When there is some level of positive correlation between planning costs and valuations this allows the firm to charge a high price to the high valuation and high planning cost consumers while still selling to some of the low valuation and low planning cost consumers.

It is worth noting here that the welfare effects of moving from the optimal single price to the optimal menu of lotteries are, in general, ambiguous. While both consumer surplus and total surplus increased in Example 1, a simple modification of the specific numbers reverses this.

**Example 2** Consider the same situation as in Example 1 but with the valuations of type B consumers increased from 20 to 30. Now the optimal single price is 30. This results in profits of 30 for the firm as well as consumer surplus of 10 for a total surplus of 40. Note that this is the maximum possible total surplus since all consumers are receiving the good with probability one. The profit maximizing mechanism, however, results in the firm offering the option of either purchasing with certainty at a price of 50 or purchasing with probability 5/7
at a price of 30. This increases profits from 30 to $35\frac{5}{7}$ but decreases consumer surplus from 10 to 0. Total surplus decreases from 40 to $35\frac{5}{7}$.

Planning costs arise naturally from the fact that time-specific goods have a single time and date at which they can be used. For example, consumers interested in purchasing an airline ticket often will need to secure time off from work at some point in advance of the travel date. If this cost is not paid (i.e. the consumer does not secure time off from work) then she will no longer have any use for a ticket. In the sale of Internet banner advertisements these planning costs appear as a result of the need for coordination and integration of advertising campaigns across various media. A company might find an Internet banner advertisement to be much more valuable if it is coordinated with a larger advertising campaign. The planning cost is explicitly the opportunity cost of planning the campaign at one time (the time of the Internet advertisement) versus another (the optimal alternative time). I note that in this case the company would still gain some value from the advertisement even if the campaign were scheduled at a different time. However, for simplicity of analysis my model will assume no utility can be derived from the good if the planning cost is not paid.

Before addressing time-specific goods directly I will consider in section three the optimal pricing mechanism for a generic good when reservation values vary with consumers’ valuations. The justification for this is that consumers with a planning cost $c$ will choose to pay this cost if and only if their expected utility from participation in the market is at least $c$. Hence, a planning cost $c$ can be modeled as a reservation utility $r = c$ for the purposes of determining the profit maximizing mechanism. My analysis will partially characterize both the situations in which a single price is and is not sufficient for maximal profits. I then return to dealing directly with planning costs and time-specific goods when I discuss implementation in section four. I show that for these goods the optimal menu of lotteries derived in section three can always be implemented by sequential price cuts with quantity restrictions at the discount prices. First I briefly review the literature.
2 Related Literature

One category of related literature addresses the implementation of mechanisms through price variations over time. I mentioned previously that there is a tendency for airline ticket prices to rise as the date of travel approaches on many if not most airline routes. Indeed, this phenomenon is well documented in the literature and there are several models that provide a range of explanations. Straightforward price discrimination is perhaps the most apparent reason as airlines seek to charge high valuation business travellers arriving late into the market higher prices by raising ticket prices as the date of travel approaches. However, Dana (1998) shows that there are also cost justifications for advance-purchase discounts and thus that rising prices could occur even in perfectly competitive markets. Gale and Holmes (1992, 1993) provides yet another explanation by showing that advance-purchase discounts on off-peak flights can efficiently induce travelers with low time costs to purchase off-peak tickets. In their models travelers have no need to plan in advance and are uncertain about their demand for a ticket and time preferences until close to the date of travel. Thus, unless a discount for advance purchase is offered, consumers have no incentive to purchase early. The model I present builds on this work by introducing the possibility that some consumers may purchase early even when not given a discount. This behavior stems from a desire to eliminate uncertainty in travel plans.

A second body of work relevant to this paper relates to the theory of auctions with endogenous entry. Gilbert and Klemperer (2000) shows that committing to a fixed price independent of the demand state can result in higher profits than adjusting prices to clear the market when consumers face entry costs. The sunk investment costs in their model play a similar role to the planning costs in my model. Their analysis, however, restricts the firm to setting a single price in any given demand state. My model differs by allowing the firm to make multiple purchasing options available to consumers which results in a descending price implementation mechanism.

A third category of recent literature relevant to this paper deals with optimal contracts
in the face of non-zero reservation values. Jullien (2000) discusses three reasons that non-zero reservation values will occur naturally in many markets: competing principals, fixed trading costs, and renegotiation. The situation in which consumers have planning costs, my primary focus in this paper, is a special instance of the fixed trading costs that Jullien discusses. However, my analysis is also partially applicable to markets in which non-zero reservation values arise from the presence of competing principals. This occurs for the following reason: when other firms are selling either the same or a similar good, purchasing from one firm means foregoing the utility that could have been achieved by purchasing elsewhere. Thus, for the purposes of determining pricing, consumers in these markets will behave as if they had reservation utility equal to the utility achievable by purchasing from another firm. However, when reservation values result from competing principals, it may not be possible to implement the optimal menu of lotteries through price cuts combined with quantity restriction as in Example 1. For this reason the implementation section is primarily targeted at situations in which the primary source of reservation utility is the presence of planning costs for consumers. Example 3 illustrates this point.

**Example 3** As in Example 1 there are two types of consumers interested in buying a time-specific good with usage date at $t = 1$. Type A consumers have valuation 100 and type B consumers have valuation 20 but now neither type has planning costs. Each group of consumers is of mass 1/2 and thus the total mass of consumers is 1. Marginal cost for the firm is zero. However, now there are numerous other firms which can produce this good at marginal cost of 50 and are willing to supply an unlimited quantity at this price. Type A consumers would choose to give up the option of buying in the outside market if and only if they can receive expected utility of at least 50 by purchasing from the firm. As a result, they behave just as the type A consumers did in Example 1 and so the optimal pricing mechanism is once again to allow consumers the choice between purchasing with certainty at a price of 50 and having a 5/8 chance of being able to buy at a price of 20.

If the option of buying from the alternate suppliers ends at some time $t < 1$ then it is
once again possible to implement the optimal mechanism by making a restricted quantity available at a discounted price at some point after the alternate suppliers close their sales. However, this will not work if it is possible to purchase from the alternate suppliers up until the date of usage. This is because cutting the price to 20 while the the alternate suppliers are still selling at 50 means that consumers who try to buy at 20 but are rationed still have the option of purchasing at 50 afterwards. As a result, if the firm tried to implement the optimal mechanism in the same way as before, then type A consumers would no longer purchase early at a price of 50 as desired. Instead they would wait and try to buy from the firm at a price of 20 and then just purchase from the outside suppliers if they happen to be denied the good.

This paper also builds off Jullien’s model of optimal contracting. While his approach allows for more generality in the structure of both production costs and demand, most of the analysis is done under restrictive conditions which, as he shows, ensure that optimal contracts will be non-stochastic. My analysis, on the other hand, is specifically directed at the case in which a stochastic mechanism is optimal.

3 The Model

A firm faces a distribution of consumers (of mass M) with valuations in the interval \([c, \bar{v}]\). Let \(F(v)\) be the cumulative distribution function (with density \(f(v)\)) that gives the mass of consumers with valuation at most equal to \(v\). For technical reasons I will assume that \(f(v) > 0\) on the interval \((c, \bar{v})\). Consumers have reservation values that vary with \(v\) and are given by \(r(v) < f(v)\). The firm seeks a mechanism that maximizes profits. According to the revelation principle it suffices to consider the optimal direct revelation mechanism in which consumers reveal their valuations. The firm assigns to each reported valuation an expected payment, \(p(v)\), and a probability of receiving the product, \(q(v)\). The consumer’s utility function, \(u(v)\) is described by \(u(v) = \max\{q(v')v - p(v')\}\). Let \(x(v)\) be defined by \(x(v) = 1\) if \(u(v) \geq r(v)\) and \(x(v) = 0\) otherwise. Thus, \(x(v)\) indicates whether consumers
of valuation \( v \) participate in the market. This assumes consumers participate whenever indifferent. The firm’s problem is given by

\[
\max \int_0^\bar{v} (p(v) - q(v)c) x(v)f(v)dv \quad (*)
\]

subject to the incentive and participation constraints:

\[
\forall v \in [0, \bar{v}] : v \in \arg \max_{v'} \{ q(v')v - p(v') \} \quad (IC)
\]

\[
\forall v \in [0, \bar{v}] : u(v) \geq r(v)x(v) \quad (PC)
\]

Before characterizing the optimal pricing mechanism in the general case, I address the simplified problem in which the reservation function, \( r(v) \), is equal to zero. I prove that maximal profits are always achievable by selling with certainty at a single price. The proof makes use of the following lemma from Myerson (1981).

**Lemma 4** When \( r(v) \) is zero, the incentive constraint (IC) is satisfied if and only if the following two conditions are met:

1. \( q(v) \) is non-decreasing in \( v \).

2. \( u(v) = \int_0^v q(u)du + u(0) = \int_0^v q(u)du \)

**Theorem 5** When \( r(v) \) is zero, the optimization problem \((*)\) always has a solution for \( q(v) \) such that \( q(v) \in \{0, 1\} \) for every \( v \in [0, \bar{v}] \).

**Proof.** The firm’s problem can be rewritten as

\[
\max \int_0^\bar{v} (p(v) - q(v)c)f(v)x(v)dv
\]

subject to the incentive and participation constraints:

\[
\forall v \in [0, \bar{v}] : v \in \arg \max_{v'} \{ q(v')v - p(v') \} \quad (IC)
\]

\[
\forall v \in [0, \bar{v}] : x(v) = 1 \text{ and } u(v) \geq 0 \quad (PC)
\]

The consumer’s payment function, \( p(v) \), can be rewritten using Lemma 4:
\[ u(v) = q(v)v - p(v) = \int_0^v q(u)du \]
\[ \Rightarrow p(v) = q(v)v - \int_0^v q(u)du \]

substituting into the firm’s objective function and integrating by parts yields

\[
\int_0^v (q(v)v - \int_0^v q(u)du - q(v)c)f(v)x(v)dv \\
= \int_0^v (q(v)(v - c) - r)f(v)dv - \int_0^v (\int_0^v q(u)du)f(v)dv \\
= \int_0^v (q(v)(v - c) - r)f(v)dv - ((\int_0^v q(u)du)F(v)) |_0^v + \int_0^v q(v)F(v)dv \\
= \int_0^v q(v)((v - c)f(v) + F(v) - M)dv \quad (**)
\]

The firm then chooses a non-decreasing function \( q(v) \) with \( 0 \leq q(v) \leq 1 \) so as to maximize (**). To complete the proof it can be shown that if the firm desires to set \( q(k) > 0 \) then it is optimal to set \( q(v) = 1 \) for all \( v \geq k \). The details are in the appendix. ■

I now consider the general case in which \( r(v) \) may vary with \( v \). The following theorems provide a partial characterization of the situations in which a menu of lotteries can achieve greater profits than are possible using a single fixed price. The need for a menu of lotteries depends on the behavior of the function \( g(v) = \frac{r(v)}{v-c} \).

**Theorem 6** When \( g(v) \) is non-increasing (*) always has a solution for \( q(v) \) such that \( q(v) \in \{0, 1\} \) for every \( v \in [0, \bar{v}] \).

**Proof.** See the appendix. ■

**Theorem 7** When \( g(v) \) is strictly increasing (*) never has a solution for \( q(v) \) such that \( q(v) \in \{0, 1\} \) for every \( v \in [0, \bar{v}] \).

**Proof.** See the appendix. ■

Thus, when \( g(v) \) is non-increasing a menu of lotteries is never necessary, but when \( g(v) \) is strictly increasing a menu of lotteries will always be required for profit maximization. In general neither of these cases can be ruled out a priori. However, when there are alternate suppliers and reservation values are equal to the expected utility of purchasing the same good from a different supplier, \( g(v) \) must be non-decreasing. To show this, let \( q_A(v) \) be the
probability of receiving the good that a utility-maximizing consumer receives when choosing to shop in the alternate market. The utility from purchasing in the alternate market for a consumer of valuation $v$ is given by $u(v) = \int_c^v q_A(u)du$. Thus, we have $g(v) = \frac{1}{v} \int_c^v q_A(u)du$ which implies $g'(v)$ is positive whenever $(v - c)q_A(u) > \int_c^v q_A(u)du$. Since $q_A(v)$ is a non-decreasing function this condition is met for any $v$ such that $q_A(v)$ is not constant on the interval $[c, v]$. The condition for Theorem 7 thus applies provided that $g(v)$ is never constant.

4 Implementation

A limiting factor in the general application of this model is the difficulty in implementing a pricing mechanism that is more complex than a single fixed price. It will almost always be impractical to explicitly allow consumers to choose from an array of lotteries. Moreover, even if this could be implemented explicitly, it would be necessary to ensure that consumers were not able to return and attempt to purchase multiple times. Otherwise consumers could select the lottery with the lowest price multiple times until they were allowed to purchase. However, as I illustrated in the previous examples, the desired effect of choosing from an array of lotteries can reasonably be implemented through a series of price drops when the good is time-specific.

To prove that this is possible in general, I consider the following two types of games. In game A a consumer chooses from an array of price probability pairs $(p_i, q_i), i = 1, ..., n$. After choosing pair $(p_i, q_i)$ the consumer has probability $q_i$ of being offered the opportunity to purchase the good at price $p_i$. Thus, a consumer with valuation $v$ will receive payoff $q_i(v - p_i)$. In game B, however, the consumer is sequentially offered a sequence of price, probability pairs, $(p_i, q_i), i = 1, ..., n$, beginning with $(p_1, q_1)$. The consumer can either accept or reject each pair in the sequence. If the consumer accepts, then with probability $q_i$ they receive the good and pay price $p_i$ as before and the game ends. However, if they reject or if they accept but don’t receive the good then the game continues and they are offered
the next pair. This continues until the consumer receives the good at some price or until all pairs have been offered. A version of game A is equivalent to a version of game B if, for any \( v \), a consumer with valuation \( v \) will have the same probability of receiving the good and will make the same expected payment when playing game B as when playing game A. To ensure uniqueness of the optimal strategy I assume that consumers always select the option that results in the highest probability of receiving the good whenever there are multiple options that result in maximal utility.

**Theorem 8** Consider a version of game A with associated lotteries \((p_i, q_i), i = 1, \ldots, n\) such that \(0 < p_1 < \ldots < p_n\) and \(0 < q_1 < \ldots < q_n \leq 1\). There exists a version of game B with associated lotteries \((p'_i, q'_i), i = 1, \ldots, n\) that is equivalent.

**Proof.** See the appendix. ■

Thus, any finite optimal array of lotteries can always be translated into a sequence of lotteries. When facing known demand, the desired sequence of lotteries can then be realized by a series of price cuts. In each price cut, the firm sets the price as desired for the corresponding lottery in the sequence but restricts the quantity sold to some fraction of the demand so as to cause rationing. While in general a continuum of purchasing options may be required for optimality, it is always possible to achieve profits arbitrarily close to the maximum with a sufficiently large but finite number of lotteries. Thus, even in this case, Theorem 8 is still sufficient to guarantee that the firm can achieve profits arbitrarily close to the maximum with a descending price mechanism.

Note that this implementation strategy depends on the rationing mechanism giving everyone who wants to purchase at the given price an equal chance of receiving the good. In practice, if the good is rationed via "first come, first serve" then this assumption is not entirely justified. For instance, it could be argued that higher valuation travelers might check prices more often and thus would be more likely to be among the first to buy after a price drop. If this is true then perhaps a better assumption is that consumers are rationed according to their valuations. However, Dana (1998) provides a counter-argument to this
point. The inevitable correlation between a consumer’s valuation and income means that higher valuation consumers will have higher time costs as well thus offsetting the higher benefit of checking prices more often. Empirical data would likely be necessary in order to determine the net effect of these competing forces.

One additional complication in the implementation of this mechanism is the possibility of resale. If the firm is unable to prevent this, then it will be optimal for third party arbitrageurs with no direct value for the good to (attempt to) purchase at the discounted price in order to resell to the rationed consumers with the highest valuations. If the resale market is efficient, then the good should always wind up in the hands of the highest valuation consumers. As such, the firm would do better to sell directly to the highest valuation consumers by raising the price until rationing no longer occurs. Thus, a descending price path which induces rationing will only be optimal when the firm can prevent resale or when resale is sufficiently costly.

5 Conclusions

When consumers have varying reservation values and/or planning costs in markets for time-specific goods, a descending price path is an optimal pricing mechanism. The optimal descending price path is characterized by a series of sudden price drops in which the firm restricts the quantity sold at all price levels excluding the highest. This allows consumers to choose between purchasing with certainty at the high price and trying to purchase at a lower price with the risk of being rationed. The greater the risk of being rationed that the consumer is willing to accept, the greater the discount attained when actually able to purchase the good. This pricing structure incents higher valuation consumers to purchase at higher prices with certainty while lower valuation consumers are still sometimes able to purchase at a discounted price. When the ratio of reservation utility to valuation is increasing as valuations increase, this makeshift price discrimination will allow the firm to
achieve greater profits than are possible by setting a single fixed price.

An important characteristic of real markets that I did not include in this model is the fact that demand is almost always uncertain to some degree. This is especially relevant for time-specific goods as firms in these markets are often forced to make quantity decisions in advance. This holds for examples such as airline and concert tickets as well as Internet banner advertising. However, by ignoring demand uncertainty, this model is able to isolate a distinct explanation for falling prices. It is possible, though, that demand uncertainty might actually aid in the implementation of this mechanism. The risk of high demand resulting in either rising prices or rationing provides an additional incentive for high valuation planners to purchase early. Thus, in real-world applications demand-uncertainty might actually reduce the amount of deliberate rationing required to incent high valuation consumers with high planning costs to purchase early.

Rationing is often a consequence of the practical inability to adjust prices quickly enough for markets to clear. However, in this model the firm intentionally lowers the price below what is necessary to sell the target quantity. The rationing that occurs is deliberate and is used as a tool to prevent high valuation consumers from waiting for prices to fall. This may explain why rationing is sometimes observed even in markets with high price flexibility. For instance, rationing is commonly seen in markets for tickets to popular concerts and sporting events. These tickets tend to sell out extremely rapidly, thus indicating that demand exceeds supply at the current price and that rationing is occurring. In such situations wealthier fans may chose to avoid this lottery by paying higher prices at a broker. While this model certainly is too simplistic to expect that every aspect be empirically accurate, the prediction that firms may use deliberate rationing to aid in price discrimination appears robust.
6 Appendix

Proof of Theorem 5

Let \( q(v) \) solve (**). I prove that either \( q(v) \) is piecewise constant or that there is another solution which is. For technical simplicity I assume that \( h(v) \) has a finite number of zeros on the interval \([0, \bar{v}]\). Partition the interval \([0, \bar{v}]\) into sub-intervals by the zeros of \( h(v) \) so that on each subinterval \( h(v) \) is either non-negative or non-positive. Consider one such interval \([a, b)\). If \( h(v) \) is non-negative on this interval then we can modify \( q(v) \) without decreasing profits by setting \( q(v) = q(b) \) for all \( v \in [a, b) \). Similarly, if \( h(v) \) is non-positive on this interval one can modify \( q(v) \) without decreasing profits by setting \( q(v) = q(a) \) for all \( v \in [a, b) \). Thus, there is a solution to (**) which is piecewise constant. Denote this solution \( q^*(v) \). Now suppose there is a subinterval over which \( q(v) = w \) for some \( w \in (0, 1) \).

Proof of Theorem 6

Let \( v^* \) be the smallest valuation such that consumers with valuation \( v^* \) have their reservation utility met and thus choose to participate. Let \( q^*(v) \) be a solution to (*). First note that \( p(v) \) is always at least \( q(v)c \). If it weren’t, the firm could increase profits by decreasing \( q^*(v) \) to zero for all such consumers. Thus, since \( u(v^*) \geq r(v^*) \) we must have that \( q(v^*)v^* - q(v^*)c \geq r(v^*) \). The incentive constraint then requires that
\(q(v^*)v - p(v^*) = q(v^*)(v - v^*) + q(v^*)v^* - p(v^*) \geq \frac{r(v^*)}{v^*-c}(v - v^*) + r(v^*) = \frac{r(v^*)}{v^*-c}(v - c)\). By assumption \(\frac{r(v)}{v-c}\) is non-increasing and so it follows that the incentive constraint requires that \(q(v^*)v - p(v^*) \geq r(v)\). This shows that the participation constraint does not bind except at \(v = v^*\). Now consider the same problem but with \(r(v) = r(v^*)\) constant. Since the participation constraint was only binding at \(v = v^*\), the solution \(q^*(v)\) must be locally optimal which is not possible unless \(q(v) \in \{0, 1\}\) for every \(v \in [0, \bar{v}]\) (by the methods used in the proof of Theorem 5). This completes the proof.

**Proof of Theorem 7**

The strategy of this proof is to derive a contradiction if such a solution exists. To do this, let \(q^*(v)\) be a solution to (*) with the property that \(q^*(v) \in \{0, 1\}\) for every \(v \in [0, \bar{v}]\) and let \(p^*(v)\) be the accompanying payment function. I will now show that the firm can offer a lottery which will be purchased by some of the consumers not currently being sold to and that this can be chosen so as not to attract any of the consumers that were already purchasing at a higher price. There exists \(v^* > c\) such that \(q^*(v^*) = 0\) if and only if \(v < v^*\). I now introduce \(q^{**}(v)\) and \(p^{**}(v)\) which are equal to \(q^*(v)\) and \(p^*(v)\) everywhere except the interval \((k, v^*)\) for some \(k \in (c, v^*)\). For \(v \in (k, v^*)\) we set \(q^{**}(v) = Q\) where \(Q\) is such that \(Q(k - c) = r(k) + \epsilon\) for \(\epsilon = \frac{1}{2}(k - c)(\frac{r(v^*)}{v^*-c} - \frac{r(k)}{k-c})\). For this same interval set \(p^{**}(v) = P = Qc + \epsilon/2\). It follows that \(Q = \frac{r(k)}{k-c} + \frac{\epsilon}{k-c} < \frac{r(v)}{v-c}\) for \(v \geq v^*\). Hence, \(Qv - P < r(v)\) for \(v \geq v^*\). This means that none of the consumers previously purchasing the good are attracted by this new lottery and so we have that \(q^{**}\) and \(p^{**}\) together satisfy the incentive constraint. Moreover, by the continuity of \(r(v)\) there is positive mass of consumers with valuations around \(k\) that are now making payments in excess of the costs of serving them. This implies that profits are higher for the firm under \(q^{**}\) which contradicts the optimality of \(q^*\).

**Proof of Theorem 8**

First note that one can throw out any of the lotteries associated with game A that are not an optimal choice for any consumer valuation. This has no effect on the expected payment
and probability of receiving the good for any consumers because these lotteries should never have been selected. Thus, without loss of generality I now assume that each of the lotteries \((p_i, q_i)\) is the optimal choice for some consumer. I now show that one can choose \((p'_i, q'_i)\) such that, in game B, playing accept beginning in stage \(i\) and in all stages thereafter results in the same probability of receiving the good and the same expected payment as selecting \((p_i, q_i)\) in game A. This follows from a simple inductive argument:

Set \((p'_n, q'_n) = (p_n, q_n)\). Now suppose that \((p'_i, q'_i)\) have been set for \(i \geq k\) such that playing accept in rounds \(i, i \geq k\), is equivalent to selecting lottery \((p_k, q_k)\). Now simply set \((p'_{k-1}, q'_{k-1})\) to satisfy the equations \(q'_{k-1}p'_{k-1} + (1 - q'_{k-1})q_kp_k = q_{k-1}p_{k-1}\) and \(q'_{k-1} + (1 - q'_{k-1})q_k = q_{k-1}\). The first equation guarantees that the expected payment from accepting beginning with round \(i - 1\) in game B yields the same expected payment as lottery \(i - 1\) in game A and the second equation guarantees that these yield identical probabilities of receiving the good. This yields \(q'_{k-1} = \frac{q_{k-1} - q_k}{1 - q_k}\) and \(p'_{k-1} = \frac{1}{q_{k-1}}(q_{k-1}p_{k-1} - (1 - q'_{k-1})q_kp_k)\). It is straightforward to check that the resulting prices are always positive and that the probabilities lie in the interval \((0, 1]\). This can be continued inductively.

I now prove that the prices, \(p'_i\), in this sequence of lotteries are strictly decreasing in \(i\) using another inductive argument. Suppose that the prices, \(p'_i\), are strictly decreasing in \(i\) for \(i \geq k\). Observe that a player will accept in round \(i\) if and only if they would prefer to buy the item for a price of \(p_i\) than play the continuation game beginning in round \(i + 1\). Thus, since prices are decreasing for \(i \geq k\) by assumption, it follows that if a player accepts in some round \(r \geq k\) then they will accept in all following rounds. The optimal strategies for the continuation game beginning in round \(k\) are consequently all of the form "accept beginning in round \(r\) and in all subsequent rounds". Thus, because of the way that \((p'_i, q'_i)\) were selected, the continuation game beginning in round \(k\) is equivalent to selecting between the lotteries \((p_i, q_i)\), \(i \geq k\), in the style of Game A.

To complete the inductive step I prove that \(p'_{k-1} > p'_k\) by contradiction. Suppose not. By assumption there is some \(v\) such that a player with valuation \(v\) selects lottery \((p_k, q_k)\) in game
A. Thus, this consumer will place accept in every round when faced with the continuation game of game B beginning in round $k$. Thus, this consumer must prefer purchasing the good at price $p_k$ to the continuation game beginning in round $k + 1$. Now note that the continuation game beginning in round $k$ will either result in this consumer purchasing the good at price $p_k$ or playing the continuation game beginning in round $k + 1$. However, if $p'_{k-1} \leq p'_k$, this consumer would select purchasing at price $p'_{k-1}$ over either of these options and thus would select purchasing at price $p'_{k-1}$ over the continuation game beginning in round $k$. From this it follows that this consumer would have selected lottery $(p_{k-1}, q_{k-1})$ over lottery $(p_k, q_k)$ in game A which contradicts our choice of this consumer. Thus, we must have $p'_{k-1} > p'_k$ and so the inductive step is now completed. This induction also establishes that the optimal strategies in game B with sequential lotteries $(p'_k, q'_k)$ are equivalent to the simultaneous lotteries offered in game A. This completes the proof of the theorem.
References


