REVENUE-MAXIMIZING MECHANISM CHOICE IN DIVISIBLE GOOD AUCTIONS: AN EMPIRICAL ANALYSIS OF THE THAI TREASURY AUCTION MARKET

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Abstract

An important question that prevails in the theory of divisible-good auctions is to determine whether a discriminatory auction yields higher revenues to the auctioneer than a uniform price auction or Vickrey auction. One way to answer this question empirically is to use estimates of the bidders’ true marginal valuations on the auctioned goods. This Honors Thesis conducts empirical studies of revenue comparisons under discriminatory and uniform price auction mechanisms and the studies of collusion on winning bid data from Thai Treasury bill auctions in 2008 and 2010. This Honors Thesis is one of the first to perform an empirical analysis using a structural econometric approach suggested by the recently developed approximate Linear Bayesian demand function equilibria (LBDFE) by [Ollikka(2011)] to estimate the model primitives and bidder valuations and simulate counterfactual auction outcomes from estimated structural parameters, then make analysis similar to methods in [Hortaçsu and Mcadams(2010)] and [Kastl(2011)]. Bank of Thailand currently employs discriminatory auctions in distributing Treasury bills. But results are generally in favor of uniform price mechanism except for those auctions with longer maturities where both auction mechanisms perform equivalently. However, practical auction experiments are necessary since the analysis modified from [Athey et al.(2011)Athey, Levin, and Seira] suggest the existence of collusion opportunities, which would not allow the uniform price mechanism to outperform the discriminatory one. Moreover, this Honors Thesis provides an indicative regression-based test for valuation implied by the data. The approaches suggested in this Honors Thesis can be extended to analyze divisible-good auctions.

**Keywords:** Multi-unit auctions, Treasury auctions, Collusion
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Chapter 1

Introduction

What is the revenue-maximizing auction mechanism for a Treasury to sell government securities? Since governments in more than 200 countries sell about $4 trillion dollars worth of securities every year ([Bartolini and Cottarelli(1994)]) and, for example, the United States issues Treasury bills in denomination of $1,000 up to a maximum purchase of $5 million annually, many economists have tried to answer this question, interpreting effectiveness from both the revenue maximization and efficiency perspectives.

The work presented here offers the empirical analysis of divisible-good auctions. In particular, the Honors Thesis presents an empirical analysis on Thai Treasury bill winning-bid data, using the recently developed structural econometric approach suggested by [Ollikka(2011)] to estimate the model primitives and bidder valuations and simulate counterfactual auction outcomes from estimated parameters. The advantage of utilizing the model of [Ollikka(2011)] is that, given the bid-level data, it provides a close-form solution to the demand equilibrium which generalizes to the multi-unit auction, apart from the fact that it is the recent literature which develops upon a wealth of auction literatures such as [Rostek et al.(2010)Rostek, Weretka, and Pycia], [Kastl(2011)] and [Hortaçsu and Mcadams(2010)], all of which carefully analyze and quantify the auction models that are applicable to divisible-good auctions. Moreover, taking each auction as a data point, I perform bootstrap resampling with replacement these auctions and re-evaluate model primitives and bidder valuations and compare the revenues under discriminatory auctions and the uniform price auctions. This gives us a better understanding of distributions of the parameter estimates.

Since the Treasury auction market in Thailand is relatively small, with less than 16 auctioneers per event, we might suspect collaborative and collusive actions among participants.
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which might know each other and place bid-price pairs in a noncompetitive manner for self
effects. Thus, based on [Athey et al.(2011)Athey, Levin, and Seira], I analyze the collu-
sion opportunities by comparing the revenues under the two auction formats for subdata
with large number of participants (≥ 11) and with small number of participants (< 11).
The reasoning behind this test is that with collusion, revenues generated by the uniform
price auction decrease because of less competition among bidders. Thus, the revenues gen-
erated by the uniform price auction and the discriminatory auction will not be significantly
different. Revenue comparisons for the entire dataset are generally in favor of uniform price
auctions since they generate significantly larger revenues, especially for those bills with short
maturities. However, results suggest the existence of collusion opportunities which would
be in favor of discriminatory auction format in terms of revenues. Therefore, experiments
are necessary before making changes in the auction format for Treasury auctions.

The outline of the Honors Thesis is as follows. In chapter 2, I provide a short literature re-
view of crucial work in auction theory and empirical analysis. In chapter 3, I present a model
of strategic bidding in a discriminatory price divisible good auction given in [Ollikka(2011)].
I then use this model to formulate optimization problem to estimate the model primitives
and bidder valuations. In chapter 4, I provide an indicative regression-based test to study
the valuation type implied by the data. Also, I apply the proposed equilibrium to estimate
the model primitives of the data on Thai Treasury auctions in 2008 and 2010, across all three
maturities (28 days, 91 days, and 182 days). Using my estimates, I conduct comparisons of
revenues between the discriminatory, and the counterfactual uniform price auctions. Next,
within the dataset, I sample with replacement the auction data and derive the distributions
for all parameter estimates. Then based on [Athey et al.(2011)Athey, Levin, and Seira], I
analyze the collusion opportunities given by the data. Chapter 5 concludes. The Appendix
A contains the derivation of the standard error approximation and the Appendix B con-
tains the empirical distributions of model estimates from bootstrap sampling of the auction
outcomes. Note that the methodologies discussed in this Honors Thesis are generalizable
to the analysis of the auctions of any divisible goods.
Chapter 2

Literature Review

In this chapter, I introduce the characterization of auction in literatures by first introducing the fundamental concepts in single-unit and multi-unit auctions. Then I discuss the celebrated Revenue Equivalence Theorem. Next, I investigate literatures that identify private and common values in the auction. The following section introduces prevailing discussions of the two pricing rules in the context of Treasury auctions. Next, I discuss recent counterfactual approaches in comparing the revenues derived from different pricing rules. Literatures on collusion are also investigated.

2.1 Single-unit Auction

The game-theoretical model of the single-unit auction suggests that strategic bidding generally fall into two model categories, a private value model and a common value model. In a private value model, each bidder assumes that each of the competing bidders obtains a random private value from a probability distribution. An example of the private value auction is the art auction where each bidder values the object differently, and independently of other people’s value. Seeing other people’s bidding would not change the behavior of one’s bidding strategy. However, in a common value model, each participant assumes that any other participant obtains a random signal from a probability distribution common to all bidders. An example of the common value model is the oil drilling right or mineral right auctions where other people’s bidding affect one’s strategies because one’s valuation for the object is correlated with other bidders’ valuation.

One way to distinguish the two models is the “winner’s curse effect,” a phenomenon that
frequently occurs in common value settings – when the actual values to the different bidders are unknown but correlated, and the bidders make bidding decisions based on estimated values. In such a case, the winner tends to be the bidder who overestimates the value of the item. A private value model assumes that the values are independent across bidders, whereas a common value model usually assumes that the values are interdependent up to the common parameters of the probability distribution.

When it is necessary to make explicit assumptions about bidders’ value distributions, most of the published research assumes symmetry, meaning that the probability distribution from which the bidders obtain their values (or signals) is identical across bidders. In a private value model which assumes independence, symmetry implies that values are independently and identically distributed. If values are not independent, they are “affiliated,” or positively correlated in a strong form. See [Milgrom and Weber(2007)].

There are traditionally four types in single-unit auction allocation, the first-price sealed-bid auction, the second-price sealed-bid auction, the open ascending auction, known as the English auction, and the open descending auction. In the first-price sealed-bid auction, bidders place their bid in a sealed envelope and simultaneously hand them in. The envelopes are opened and the individual with the highest bid wins, paying a price equal to the exact amount that one bids. In the second-price sealed-bid auction, bidders place their bid in a sealed envelope and hand them in, except that the winners pay the price equal to the exact amount of the second highest bid. In the English auction, price is steadily raised with the bidders dropping out once the price becomes too high, in most cases higher than some people’s valuation for the object. The auction process continues until there remains only one bidder who wins the auction at the current price. The last type is the open descending-bid auction (Dutch auctions) in which the price starts at a level sufficiently high to deter all bidders and is progressively lowered until a bidder indicates that he is prepared to buy at the current price. He or she wins the auction and pays the price at which they bid.

2.2 Multi-unit Auction

Examples of multi-unit auctions include the Treasury bill auction where bidders submit price-quantity pairs of bills and the flowers or wines auctions. Multiple objects can be substitutes or complements or identical such as the Treasury bills. According to [Krishna(2010)], there are three sealed-bid auctions for selling identical units: (i) the discriminatory auction,
(ii) the uniform auction, and (iii) the Vickrey auction. For all auction formats, bidders who value the object the most win the object.

In a discriminatory auction, each bidder pays an amount equal to the sum of his winning bids, and a bidder pays exactly what he bids. In a uniform price auction, all units are sold at a market-clearing price such that the total amount demanded is equal to the total amount supplied. The market-clearing price is the highest losing bid. In a Vickrey auction, a bidder who wins $k^i$ units pays the $k^i$ highest losing bids of the other bidders not including his own.

Assume that auctions are in the private value environment. The Vickrey auction is an appropriate extension of the second-price sealed-bid auction. In fact, the terms Vickrey auction and the second-price sealed-bid auction are used interchangeably, because in both cases bidders submit the bid without knowing the bid of other people and the winner is the one who bids the highest but the price paid is equal to the second highest price. Moreover, unlike the uniform price auction, both the Vickrey auction and the discriminatory auction give bidders incentives to bid their true value. The uniform price auction generalizes to the first-price sealed-bid auction in single-unit case because both encourage bid shading.

### 2.3 Revenue Equivalence Theorem

One of the main findings of the auction theory is the celebrated revenue equivalence theorem first proved by [Vickrey(1961)]. The theorem states that any allocation mechanism with all of the following properties leads to the same expected revenue for the seller: any player under any type of auction can expect the same surplus if (i) the bidder with the highest type or signal or valuation always wins; (ii) the bidder with the lowest possible type or signal or valuation expects zero surplus; (iii) all bidders are risk neutral; (iv) all bidders are drawn from a strictly increasing and atomless distribution.

**Risk Averse Environment** Suppose the bidders are risk averse and have constant absolute risk aversion, then the following holds: (i) in the second-price and English auctions, revealing public information increases the expected price; (ii) among all possible information reporting policies for the seller in second-price and English auctions, full reporting leads to the highest expected price; (iii) the expected price in the English auction is at least as large as in the second-price auction. In practice we cannot satisfy all conditions simultaneously. Under independent values and risk aversion, the first-price auction leads to higher prices
than the second-price auction. In conjunction with earlier results, first-price and second-price auctions that include both affiliation and risk aversion cannot generally be ranked by their expected prices. Releasing information can reduce the risk premium demanded by the bidders. Although the risk premium can be reduced to zero by releasing perfect information, it is not generally true that partially resolving uncertainty, for example by releasing incomplete information, will reduce the risk premium.

The revenue equivalence theorem is not limited to the single-unit auction, but also extends to the multi-unit auction, with similar arguments to above.

### 2.4 Value Identification

In order to compare revenue, it is helpful to identify the valuation types in the auction data. [Milgrom and Weber(2007)] introduce the theory of equilibrium bidding in different auction environments. For single-unit auctions, [Gilley and Karels(1981)] were the first to propose a reduced form testing approach based on examining how bids vary with the number of participants. For second-price sealed-bid and English auctions, [Hong and Shum(2000)] and [Bajari and Hortacsu(2003)] employ tests for common value using standard regression techniques: under the maintained null hypothesis of independent private values and (weakly dominant strategy) truthful bidding, bids should not respond to information about the number of participants. [Pinkse and Tan(2005)] establish, however, that such a reduced form test cannot distinguish unambiguously a common value from an affiliated private value model in first-price auctions. [Hong and Shum(2000)] showed that a more detailed structural model could achieve the goal of distinguishing common value from private value in first-price auctions. [Haile et al.(2006)] Haile, Hong, and Shum] develop a nonparametric test for common value in first-price auctions making use of variation in the number of bidders across auctions. This nonparametric test is based on [Laffont et al.(1995)Laffont, Ossard, and Vuong] and [Guerre et al.(2000)Guerre, Perrigne, and Vuong] to estimate the distribution of valuations given the observed bids. In particular, their theory predicts a specific ordering between the distribution of the expected value of the object conditional on winning under common valuation paradigm as the number of bidders varies, while the distribution should not vary with the number of participants under private value.
2.5 Treasury Auctions Under Discriminatory and Uniform Price Auctions

For Treasury auctions, there are two possible factors that can lead to different expected revenue among the two pricing mechanisms: discriminatory pricing and uniform pricing. The first factor is downward bias in bids. The Treasury auction is generally a common values auction in which auctioneers hold similar beliefs about values of the bills but they are likely to have different information about the \textit{ex ante} value. This influences the outcome of the auction in a crucial way. A bidder that wins the auction by placing the highest bid is effectively making an above-average assessment of the resale price (or “true” value) of the security, a “winner’s curse effect” that raises the likelihood of a loss in the post-auction market, resulting in auctioneers shading their bids downward relative to their subjective assessment of the security’s resale value to counter the risk of incurring a loss in the secondary market for the security. Another factor is the vulnerability to noncompetitive bidding behavior. Lack of competition is likely to discourage participation in the Treasury bill auction and restrict a security’s supply in the secondary market, preventing that security’s efficient allocation among investors. However, this is not of our concern since the Thai Treasury auction does not have a noncompetitive billing component.

The question then arises: Which of the two methods is likely to cause participants to shade bids downward to a greater extent? The answer is the discriminatory auctions, because in this format winners pay their own bid, they are charged fully for their errors in overseeing the object’s resale value. In contrast, in uniform price auctions winners pay a price close to the highest losing bid. This price reflects, to some extent, other bidders’ view of the object’s value. This feature lowers the winners’ risk of paying a price that far exceeds consensus and hence encourages more aggressive bidding. Thus, as a means of maximizing revenues, the uniform price method is likely to outperform the discriminatory method in Treasury auctions because it is less vulnerable to the winner’s curse.

Some of the empirical analyses are in favor of the uniform price auction. According to [Bartolini and Cottarelli(1994)], since introducing auctions for bills in 1929, the United States Treasury has used a discriminatory rule to auction most of its securities, except for a short period in 1973-1974 when uniform price auctions were held for long-term securities as an experiment, during which the revenues were increased by 0.75 percent. This evidence remains controversial, however, because it relies largely on data from the August 1973
auction, which was undersubscribed and which awarded a large portion of the security to United States government accounts. Nevertheless, other evidence supports the finding the uniform price auction leads to a rise in revenues. [Umlauf(1993)], for instance, studies the Mexican Treasury’s experience with uniform price auctions from 1990 to 1993. Comparing returns from these auctions with returns from discriminatory auctions held from 1986 to 1991, he finds evidence of higher revenues from the uniform pricing mechanism.

[Milgrom and Weber(2007)] and [Ritter(1960)] (Friedman) argue for uniform pricing, in part because they believe the common values plays more important role in the Treasury auctions and because it encourages more participation than the discriminatory auction. Moreover, according to [McAfee and McMillan(1987)] and [Milgrom and Weber(2007)], under risk neutrality, the common value auction theory with affiliated information predicts that the uniform pricing would alleviate the winner’s curse, and thus would encourage bidders to bid more aggressively. The uniform price auction leads to a better distribution of auction awards. Under this system, bidders bid more aggressively without fear of the winner’s curse. This is because they will get the securities issued at the price quoted by the lowest accepted bid and not the actual that they have bid as in the discriminatory auctions. Hence, uniform price auctions are expected to enhance market efficiency.

Overall, empirical research on auctions argue for the uniform pricing rule for auctioning the Treasury bills. However, according to [Bartolini and Cottarelli(1997)], out of 42 countries that use auction to sell Treasury bills, Denmark and Nigeria were the only two countries using uniform price auctions. Spain used a mixed format in which bidders posting less than average bids were charged their bid, while remaining bidders were charged a uniform average bid. Interestingly, six countries – Belgium, France, Italy, Gambia, Mexico, and Tanzania – had used uniform price auctions in the past but returned to discriminatory auctions before the survey was taken. However, there is no occurrence of permanent shift from discriminatory to uniform price auctions. It is worth investigating why most countries do not prefer uniform price auction format despite being less vulnerable to the winner’s curse effect. The revenue comparison among these two pricing rules is currently ambiguous and is open to empirical research.
2.6 Counterfactual Analysis

Modern strategies to conduct counterfactual analysis for revenue comparison is based on the seminal work of [Wilson(1979)]. [Hortaçsu and Mcadams(2010)] propose an estimation method to bound bidders’ marginal valuations in discriminatory auctions using individual bid-level data, and apply the method to data from Turkish Treasury auction market. The paper concludes that switching from a discriminatory auction to a uniform price or Vickrey auction would not significantly increase revenue. [Kastl(2011)] provides an important critique of the bounding strategy given in [Hortaçsu and Mcadams(2010)] by showing that when there are constraints on the number of price-quantity bids that a bidder can place in a given auction, it may be optimal for bids to exceed marginal valuations in a uniform price auction. [Kastl(2011)] concludes that the uniform price auction performs well, both in terms of efficiency of the allocation and in terms of revenue maximization.

My revenue comparison analysis is based on the model of [Ollikka(2011)], which provides a parametric approach to analyze the multi-unit auction mechanism in the environment of incomplete information. [Ollikka(2011)] extends the existing model of [Rostek et al.(2010)] which is based on specified conditions of symmetric bidders and full information. Moreover, the paper constructs a demand function model for a Vickrey auction, a uniform price auction, and a discriminatory auction. The solution concept is based on symmetric linear Bayesian demand function equilibria (LBDFE) and it is building on an affine information structure similar to [Vives(2009)] and [Vives(2011)]. [Ollikka(2011)] builds upon the solution concept by allowing for uncertainty in the quantity to be auctioned. To be more precise, the paper extends existing research by studying the equilibrium of the multiunit auction under uncertain and correlated values. [Vives(2011)] establishes a model of uniform price auction, which allows for common and private values in the absence of exogenous noise, while [Ollikka(2011)] allows for uncertain and possibly correlated valuations.

2.7 Collusion in Auction Market

Most auctions, including Treasury auctions, are vulnerable to collusion among the bidders. Since the presence of collusion is detrimental to the final price and can potentially result in a significant reduction in revenue gain, it is useful for the auction issuer to be able to detect a concerted action among the ring of bidders.

To test for collusion in the discriminatory auction, we first consider collusion in the
first-price sealed-bid auction. Even if bidders are *ex ante* symmetric, collusion in the first-price sealed-bid auctions introduces asymmetries among bidders, resulting in inconclusive bidding behavior. Under both the presence of collusion and the common values, the revenue comparison among the two auction formats of the Treasury auction is inconclusive: the common values auctions tend to favor the uniform auction, while the asymmetries, which arise from the collusion in the first-price sealed-bid auctions among bidders, tend to favor the discriminatory auction. Again, it is left to both empirical and theoretical work to investigate whether the collusion effect or the asymmetry effect plays a more significant role in revenue determination.

Chapter 3

Methodology

3.1 Data and Institutional Background

Since 1945, with a 10-year pause from 1990 to 1999, the Bank of Thailand issues weekly Treasury bills through discriminatory auctions. Although Treasury bills in most countries are issued both through a competitive bidding process at a discount from par and a non-competitive bidding, Thai Treasury does not issue noncompetitive bills. Our data consist of all winning bids in 2008 and 2010. There are three types of times to maturities ($T$): $T = 28$ days, $T = 91$ days, and $T = 182$ days. Bids are submitted electronically, and cannot be observed by other participants. Bidders can revise their bids until the submission deadline. The minimum bid is 10 million baht and the minimum bid increment is 1 million baht. Bidders are allowed to submit up to 3 bid points (price-quantity pairs) in any given function. Those quantities with lowest proposed prices that are lower than reserve prices win up to the quota limit for each auction. There exists a secret reserve price policy, but according to the data, most winning yields are accepted. Unlike Treasury auctions in other countries, there is no noncompetitive bidding component. Please note that the summary statistics is not provided for privacy purposes.

According to the information given by the Bank of Thailand, the formula for calculating the price is given by

$$ price = \frac{F}{1 + \left( \frac{Y}{100} \cdot \frac{d}{365} \right) } $$

where $F$ is the face value of the Treasury bill, $Y$ is the simple annual yield, $d$ is the remaining
days to maturity. We can easily see that the price that the bidders pay must be less than the face value of the bill. For example, if the bill is worth 100 million baht at maturity date, the bidder might pay only 98 million baht at the beginning and hold on to the bill until end of maturity when the bidder receives 100 million baht. Minimum bid is 10 million baht and minimum bid increment is 1 million baht.

3.2 Theoretical Auction Model and Optimization Problem

According to [Ollikka(2011)], the model is applicable to auctions with more than three risk neutral bidders. Assume that for each bidder $i$, the marginal value function is linear in quantity,

$$u_i(q_i) = \theta_i - \beta q_i$$

where $\theta_i$ represents the value parameter and $\beta$ is the slope of the demand schedule, which is constant for all bidders. Each bidder’s demand schedule maximizes his or her expected profit. The demand schedule is assumed to be linear in the signal of the value and price:

$$D_i(s_i, p) = a + bs_i - cp$$

where $a, b, c$ are positive constants and $s_i$ is the signal of the value parameter $\theta_i$ received by bidder $i$. We let $Q$ denote the total quantity supply in the auction. [Ollikka(2011)] assumes $Q$ to be random, but in our case we take $Q$ to be fixed and known.

In the multi-unit auction, bidders submit non-increasing bid schedules $D_i(p)$. Then, the seller determines an aggregate demand $D(p) = \sum_{i=1}^{k} D_i(p)$, where $k$ is the number of bidders whose bids are accepted, and equalizes demand and supply $Q = D(p^*)$, where $p^*$ is a clearing price. Thus, every bidder faces a residual supply curve $S_i(p) = Q - \sum_{j\neq i}^{k} D_j(p)$. All demands above the clearing price are accepted.

The value parameter of the linear marginal value function $\theta_i$ is normally distributed, $\theta_i \sim N(\bar{\theta}, \sigma^2_{\theta})$. Value parameters are correlated between bidders with $\text{cov}(\theta_i, \theta_j) = \rho \sigma^2_{\theta}$. An agent receives a signal $s_i = \theta_i + \varepsilon_i$, where $\varepsilon_i$ are independently and identically distributed with $\varepsilon_i \sim N(0, \sigma^2_{\varepsilon})$. The model is a pure private values model if signals are perfect ($\sigma^2_{\varepsilon} = 0$) or there is no correlation between the value parameters ($\rho = 0$). If however, $0 < \sigma^2_{\varepsilon} < \infty$ and $0 < \rho < 1$, bidders have both private and common values. If $\rho = 1$, the model is pure common values. Bidders do not observe $\bar{\theta}$ and $\varepsilon_i's$ separately, but instead they
observe \( s_i = \theta_i + \varepsilon_i \). Let \( n \) be the number of bidders. Then, the average signal is given by 
\[
\bar{s} = \frac{1}{n} \sum_{i=1}^{n} s_i \quad \text{where} \quad \text{E} [\bar{s}] = \bar{\theta} \quad \text{and} \quad \text{Var} [\bar{s}] = \frac{1}{n} \left( \sigma^2 + (1 - \rho + n\rho) \sigma^2 \right).
\]

According to [Ollikka(2011)], the expected value of the marginal value function \( \theta_i \) conditional on a signal \( s_i \) and on a price \( p \) is given by

\[
\begin{align*}
\text{E} [\theta_i | s_i, p] &= A \theta_i + B s_i + C \frac{1}{b} \left[ Q - (n - 1) a + (n - 1) cp - q_i \right], \\
A &= \frac{\sigma^2_\varepsilon \left( \sigma^2 + (1 - \rho) \sigma^2_\theta \right)}{\sigma^2_\varepsilon + (1 - \rho) \sigma^2_\theta} \left( \sigma^2 + (1 - \rho + n\rho) \sigma^2_\theta \right), \\
B &= \frac{\sigma^2_\theta \left( \sigma^2_\varepsilon + (1 - \rho) \sigma^2_\theta \right)}{\sigma^2_\varepsilon + (1 - \rho) \sigma^2_\theta} \left( \sigma^2 + (1 - \rho + n\rho) \sigma^2_\theta \right), \\
C &= \frac{\rho \sigma^2_\varepsilon \sigma^2_\theta}{\sigma^2_\varepsilon + (1 - \rho + n\rho) \sigma^2_\theta}, \\
M &= \frac{n \rho \sigma^2_\varepsilon}{(1 - \rho) \left( \sigma^2_\varepsilon + (1 - \rho + n\rho) \sigma^2_\theta \right)}, \\
a &= \left( \frac{M}{1 + M} \right) \frac{Q}{n} \frac{1}{\beta} \left( \frac{1}{1 + M} \right) \frac{A \theta_i}{b} , \\
b &= \frac{1}{\beta} \left( B - C \right), \\
c &= \frac{1}{\beta} \left( \frac{1}{1 + M} \right).
\end{align*}
\]

In order for strategies to be linear in the discriminatory auction, the distribution of \( Q \) must have a linear and non-increasing inverse hazard rate, which is defined by \( \lambda (Q) = \frac{1 - F (Q)}{F (Q)} \), where \( F (Q) \) is the cumulative distribution function and \( f (Q) \) is the density function of the total quantity \( Q \). [Ollikka(2011)] assumes that this condition is automatically satisfied, so the condition holds for the case where \( Q \) is fixed.

The equilibrium strategy in the discriminatory auction where \( Q \) is fixed is defined by

\[
D_i^q (s_i, p) = \left( \frac{M}{1 + M} \right) \frac{Q}{n} \frac{1}{\beta} \frac{1 - \rho}{\sigma^2_\theta} \left( \frac{1 - \rho + n\rho}{\sigma^2_\theta} \right) s_i + \frac{1}{\beta} \left( \frac{1}{1 + M} \right) \left( A \theta_i - p \right).
\]

This is equivalent to writing the expected demand equation as

\[
E [p] = a_0 (n, Q) + a_1 (n) q_i,
\]
where

\[ \alpha_0(n, Q) = \theta \sigma^2_\varepsilon + \frac{\rho \sigma^2_\varepsilon Q}{n \beta (1 - \rho) \text{Var}[\tilde{s}]} - \frac{(1 - \rho) \sigma^2_\theta}{\sigma^2_\varepsilon + (1 - \rho) \sigma^2_\theta} \left( 1 + \frac{\rho \sigma^2_\varepsilon}{(1 - \rho) \text{Var}[\tilde{s}]} \right) \]

and

\[ \alpha_1 = -\beta \left( 1 + \frac{\rho \sigma^2_\varepsilon}{(1 - \rho) \text{Var}[\tilde{s}]} \right) \]

In order to carry out the counterfactual analysis, we need to identify model estimates \( \hat{\theta}, \hat{\beta}, \hat{\sigma^2_\theta}, \hat{\sigma^2_\varepsilon}, \hat{\rho} \). One way to estimate these parameters is to solve the following unconstrained optimization problem:

\[ \text{minimize } \sum_{i=1}^{N} (\alpha_0(n_i, Q_i) + \alpha_1(n_i) q_i - p_i(q_i))^2 \]

with variables \( \alpha_0 \) and \( \alpha_1 \) and \( N \) is the number of auctions in a year for each maturity. Solving this optimization yields

\[ \hat{\alpha}_1 = \frac{\bar{p} \bar{q} - \bar{p} \bar{q}}{\bar{q}^2 - (\bar{q})^2}, \quad \hat{\alpha}_0 = \bar{p} - \hat{\alpha}_1 \bar{q} \]

where \( \bar{p} \bar{q} = \frac{1}{N} \sum_{i=1}^{N} p_i q_i, \bar{p} = \frac{1}{N} \sum_{i=1}^{N} p_i, \bar{q} = \frac{1}{N} \sum_{i=1}^{N} q_i \) and \( \bar{q}^2 = \frac{1}{N} \sum_{i=1}^{N} q_i^2 \). Given \( \hat{\alpha}_0 \) and \( \hat{\alpha}_1 \), we derive the parameter estimates \( \hat{\theta}, \hat{\beta}, \hat{\sigma^2_\theta}, \hat{\sigma^2_\varepsilon}, \hat{\rho} \) by solving an optimization problem

\[ \text{minimize } \sum_{i=1}^{n} (\alpha_0(\tilde{\theta}, \tilde{\beta}, \tilde{\sigma^2_\theta}, \tilde{\sigma^2_\varepsilon}, \rho, n_i, Q_i) - \hat{\alpha}_0(i))^2 + (\alpha_1(\tilde{\theta}, \tilde{\beta}, \tilde{\sigma^2_\theta}, \tilde{\sigma^2_\varepsilon}, \rho, n_i) - \hat{\alpha}_1(i))^2 \]

subject to \( \sigma^2_\theta, \sigma^2_\varepsilon \geq 0, \quad 0 \leq \rho \leq 1 \)

Consider two optimization approaches. One is the simulated annealing. Another is the grid search over each variable to obtain a set of locally optimal parameters.

### 3.3 Optimization Methodology

The two main optimization approaches are implemented: simulated annealing and coordinated descent with grid search.
3.3.1 Simulated Annealing

Simulated annealing is used to solve the optimization problem since it can locate a good approximation to the global optimum of a given function in a large search space. The name and inspiration come from annealing in metallurgy, a technique involving heating and controlled cooling of a material to increase the size of its crystals and reduce their defects.

We start by constructing the sample from solutions near the current solution from a candidate distribution (proposal distribution, which in our case is a standard Gaussian distribution). Then the new solution may be accepted with a probability that depends both on the difference between the corresponding function values and on a global parameter $T$ (called the temperature), that is gradually decreased during the process. The dependency is such that the the choice between the previous and current solution is almost random when $T$ is large, but increasingly selects the better or “downhill” solution (for a minimization problem) as $T \to 0$. This method is clearly an adaptation of the Metropolis-Hastings algorithm, a Monte Carlo method to generate sample states of a thermodynamic system.

In other words, at each step, the simulated annealing considers some neighboring states $s'$ of the current state $s$, and probabilistically decides between moving the system to state $s'$ or staying in state $s$. These probabilities ultimately lead the system to move to states of lower energy. Typically this step is repeated until the system reaches a state that is good enough for the application, or until a computation budget has been exhausted. Searching for neighbors of a state is fundamental to optimization because the final solution will come after a tour of successive behaviors.

**Transition Probabilities:** For each edge $(s, s')$ of the search graph, one defines a transition probability, which is the probability that the simulated annealing algorithm will move to state $s'$ when its current state is $s$. This probability depends on the current temperature, by the order in which the candidate moves, and by the acceptance probability function.

**Acceptance Probabilities:** Most papers define the acceptance probability function as follows:

$$P (e, e', T) = \begin{cases} 1 & \text{if } e' < e \\ \exp ((e - e') / T) & \text{otherwise} \end{cases}$$
This corresponds to the Metropolis-Hastings algorithm, in case where the proposal distribution is symmetric. However, \( P(e,e',T) \) is often used for simulated annealing even if the proposal distribution in Metropolis-Hastings is not symmetric, or not probabilistic (the probability density function does not integrate to 1). We begin the first few iterations with \( T \) being a large number, so we accept almost all moves with probability close to 1. However, for later iterations, when \( T \) gets smaller, we accept with less probability so that after we allow function to explore various optima, we constrain the function to a smaller number of possible states.

Implementation of this optimization scheme is currently in progress. So far the results are not superior to the results from the grid search method discussed next. Factors that are needed in order to improve the results from this algorithm are such as appropriate cooling schedule and correct initializations for a stationary sampling.

### 3.3.2 Coordinate Descent with Grid Search

Remember that we are solving a constrained optimization with inequality constraints specified in the previous section. Gradient-based method such as line search does not yield satisfying results. Moreover, since the function is sensitive to parameters, introducing gradients can introduce more singularities. Also, given an intractable analytic derivative expressions, approximating gradients by finite difference method introduces more sources of randomness. The objective function is not convex in parameters of interest. These reasons convince us to consider using the non-derivative optimization algorithm such as the coordinate descent method with grid search in the feasible parameter space over random initialization in order to obtain local minima.

For optimization mechanism, I consult [Press et al. (2007)](Press, Teukolsky, Vetterling, and Flannery). I do grid search along one coordinate direction at the given starting point in each iteration, then apply this procedure with other coordinate directions cyclically throughout. In fact, given the existence of the gradients, iterations of a cycle of line search in all directions is equivalent to one gradient descent iteration.

Suppose we start by minimizing over the \( x \) – coordinate direction and we start at the point \( x = x_0 \). Then perform grid search search along this direction within the interval \((x_0 - \delta, x_0 + \delta)\), where \( \delta > 0 \) is a fixed radius of the interval. Grid search considers the value \( x_1 \in (x_0 - \delta, x_0 + \delta) \) that yields the lowest objective in this interval. Next, we initialize the next iteration at \( x_1 \) and do grid search within the interval \((x_1 - \delta, x_1 + \delta)\), and from
this iteration we will get the value $x_2$ that minimizes the objective. Perform this search repeatedly and choose the iteration that yields $x^*$ associated with the lowest objective function. Update the argument in the objective function by replacing $x$ with $x^*$, and repeat this process for all other directions.

For each direction of coordinate descent, we can further anneal this grid search by shrinking the radius of the search interval at each iteration. The motivation for annealing is that for iteration $i$ along the $x$-coordinate, we only search for $x$ that minimizes the objective from the candidates in the set

$$\left\{ x_{i-1} - \delta, x_{i-1} - \delta + \frac{2\delta}{N}, x_{i-1} - \delta + \frac{4\delta}{N}, \ldots, x_{i-1} + \delta \right\}$$

which contains $N+1$ candidates for a given $N > 0$, the number that specifies the resolution of grid search. Since we only have finitely many candidates $x$ in the set that we search over and particularly in our case where the function is not smooth and is sensitive to parameters, we want to perform a fine-tune search for each coordinate. For example, consider shrinking the search interval for each subsequent iterations by defining the sequence $\{\delta_i\}_{i \in \mathbb{N}}$ to be

$$\delta_i = \delta e^{-\frac{i}{\beta}}$$

for iteration $i$ where $\beta > 0$ is a constant. Substitute this $\delta_i$ into the procedure described earlier at each iteration $i$. For iteration $i - 1$, the coordinate descent will search for the $x_i \in (x_{i-1} - \delta_i, x_{i-1} + \delta_i)$ that minimizes the objective, and we update $x$ with $x_i$ and perform similar process to all other coordinate directions. The lower the value of $\beta$, the faster the search interval shrinks. Note that for iteration $i$, we search for $x$ that minimizes the objective function from the candidates in the set

$$\left\{ x_{i-1} - \delta_i, x_{i-1} - \delta_i + \frac{2\delta}{N}, x_{i-1} - \delta_i + \frac{4\delta}{N}, \ldots, x_{i-1} + \delta_i \right\}$$

which contains $N+1$ candidates for a given $N > 0$ similar to the previous set. However, for the later iterations, $\delta_i \to 0$ as $i \to \infty$, so we are still searching over $N + 1$ candidates of $x$’s in each iteration but from a smaller and smaller search interval so that the search is more efficient. Results are given in the next section.
Chapter 4

Result and Counterfactual Analysis

4.1 Value Identification: The Regression Approach

Since [Ollikka(2011)] assumes the use of linear strategies and the data is not linear, I investigate the type of valuation (private values and/or common values) by fitting or calibrating the data using linear regression approach. We are interested in the effect of number of bidders \((n)\) on price \((p)\), standard deviation of the price \((\sigma_p)\), and coefficient of variation in the price \((c_p)\). The dependent variables are determined by the following formulas:

\[
p = \frac{\sum_{i=1}^{n} p_i q_i}{\sum_{i=1}^{n} q_i}, \quad \sigma_p = \sqrt{\frac{\sum_{i=1}^{n} (p_i - p)^2 q_i}{\sum_{i=1}^{n} q_i}}, \quad c_p = \frac{\sigma_p}{p}
\]

where \(q\) is the quantity bid. The regression results are as follows:

<table>
<thead>
<tr>
<th>Maturity ((T))</th>
<th>Regression</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 days</td>
<td>(y = 2.981 + 0.0155x) ((0.2173) \quad (0.0200))</td>
</tr>
<tr>
<td>91 days</td>
<td>(y = 2.6854 + 0.0423x) ((0.2353) \quad (0.0212))</td>
</tr>
<tr>
<td>182 days</td>
<td>(y = 3.3383 - 0.0091x) ((0.2033) \quad (0.0177))</td>
</tr>
</tbody>
</table>

Table 4.1: Regression results of \(y = p\) on \(x = n\), 2008 T-Bill
### CHAPTER 4. RESULT AND COUNTERFACTUAL ANALYSIS

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>Regression</th>
</tr>
</thead>
</table>
| 28 days      | $y = 0.7662 + 0.0311x$  
(0.1845) (0.0135) |
| 91 days      | $y = 0.9196 + 0.0236x$  
(0.2542) (0.00198) |
| 182 days     | $y = 1.6159 - 0.0128x$  
(0.1519) (0.0106) |

Table 4.2: Regression results of $y = p$ on $x = n$, 2010 T-Bill

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>Regression</th>
</tr>
</thead>
</table>
| 28 days      | $y = 0.0384 - 0.0027x$  
(0.0084) (0.0008) |
| 91 days      | $y = 0.0225 - 0.0009x$  
(0.0109) (0.0010) |
| 182 days     | $y = 0.0567 - 0.0038x$  
(0.0090) (0.0008) |

Table 4.3: Regression results of $y = \sigma_p$ on $x = n$, 2008 T-Bill

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>Regression</th>
</tr>
</thead>
</table>
| 28 days      | $y = 0.0097 - 0.0001x$  
(0.0087) (0.0006) |
| 91 days      | $y = 0.3508 - 0.0225x$  
(0.2536) (0.0198) |
| 182 days     | $y = 0.0203 - 0.0008x$  
(0.0062) (0.0004) |

Table 4.4: Regression results of $y = \sigma_p$ on $x = n$, 2010 T-Bill
In a competitive market, as the number of bidders increases, both the price and the standard deviation of the price should increase. Most of the results from regressing $p$ on $n$ and $\sigma_p$ on $n$ indicate negative slopes, which suggest that the market is not competitive and bidders are more afraid of the winner’s curse as the number of bidders increase. Note that the $t-$statistics of the coefficients of $\sigma_p$ are all significantly different from 0 at $\alpha = 5\%$, and the $t-$statistics of the coefficients of $p$ are significantly different from 0 for 91-day maturity of 2008 Treasury bill and both 28- and 91-day maturities of 2010 Treasury bill, suggesting the strong winner’s curse effect. The slopes from regressing $c_p$ on $n$ are also negative. Observe that the $t-$statistics of the coefficients of $c_p$ are increasingly significant as the time to maturity $T$ increases. This suggests that the standardized ranges of bid values decrease as the number of bidders increases. In other words, as the number of bidders increase, the variation in price reduces faster than the price level, suggesting that bidders bid much less aggressively due to the winner’s curse effect. Regression results suggest the presence of the winner’s curse which implies that Thai Treasury auctions are increasingly dominated by the common values, as time to maturity increases.
4.2 Parameter Estimates

Grid search yields the following parameter estimates. Bootstrap standard errors of these estimates are given in the parentheses. See more details in Appendix B.

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>Objective</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{\sigma}_\theta^2$</th>
<th>$\hat{\sigma}_\epsilon^2$</th>
<th>$\hat{\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 days</td>
<td>2.3509</td>
<td>0.1572</td>
<td>112.6289</td>
<td>0.4143</td>
<td>0.8710</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0400)</td>
<td>(24.8054)</td>
<td>(8.9823)</td>
<td>(0.3295)</td>
<td></td>
</tr>
<tr>
<td>91 days</td>
<td>3.0868</td>
<td>0.9169</td>
<td>129.4591</td>
<td>0.2522</td>
<td>0.8847</td>
<td>0.1541</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0656)</td>
<td>(33.9809)</td>
<td>(9.2504)</td>
<td>(0.3108)</td>
<td></td>
</tr>
<tr>
<td>182 days</td>
<td>3.1558</td>
<td>0.2067</td>
<td>−58.1478</td>
<td>4.0707</td>
<td>0.0126</td>
<td>0.9690</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.1712)</td>
<td>(34.2432)</td>
<td>(9.5547)</td>
<td>(0.2972)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.7: Minimized objectives and associated parameter estimates of 2008 T-Bill

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>Objective</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{\sigma}_\theta^2$</th>
<th>$\hat{\sigma}_\epsilon^2$</th>
<th>$\hat{\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 days</td>
<td>1.3894</td>
<td>0.5959</td>
<td>110.8987</td>
<td>0.4804</td>
<td>0.9312</td>
<td>0.1184</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.1000)</td>
<td>(31.4788)</td>
<td>(10.6300)</td>
<td>(0.8349)</td>
<td></td>
</tr>
<tr>
<td>91 days</td>
<td>0.8814</td>
<td>0.1498</td>
<td>−11.9682</td>
<td>0.3746</td>
<td>2.0444</td>
<td>0.2702</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.1432)</td>
<td>(23.5614)</td>
<td>(8.4008)</td>
<td>(0.2834)</td>
<td></td>
</tr>
<tr>
<td>182 days</td>
<td>1.0974</td>
<td>0.0165</td>
<td>8.7987</td>
<td>0.6273</td>
<td>0.7162</td>
<td>0.0350</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.1192)</td>
<td>(28.3785)</td>
<td>(13.6774)</td>
<td>(0.3320)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8: Minimized objectives and associated parameter estimates of 2010 T-Bill

It is difficult to make exact interpretations from parameter estimates since the relationships between parameters and maturities and relationships among parameters are mostly unknown. Observe that in general $\hat{\sigma}^2_\theta$ increases with maturities, suggesting that bidders are more uncertain in estimating their own valuation. Moreover, according to bootstrap sampling results in Appendix B, we observe a higher empirical mean of $\hat{\sigma}^2_\theta$ as time to maturities increases: for 2008 T-Bills, the empirical mean of $\hat{\sigma}^2_\theta$ is $E[\hat{\sigma}^2_\theta] = 8.1241, 8.0181$ for 28-day and 91-day maturities respectively, and $E[\hat{\sigma}^2_\theta]$ increases to 8.6838 for 182-day maturities; for 2010 T-Bills, we observe an obvious increasing trend of $E[\hat{\sigma}^2_\theta]$ to be 6.9588, 8.4606 and 9.8702 for 28-day, 91-day, and 182-day maturities respectively.

The auction is pure private value if signals are perfect, $\hat{\sigma}^2_\epsilon = 0$ or there is no correlation between the value parameters, $\rho = 0$. We see that the only auction that appears to be the pure private values auction is the one in 2008 with 28-days maturity, and others contain the
common values component. Both variables quantify common values component: \( \hat{\rho} \) should increase with time to maturities as common values component dominates and \( \hat{\sigma}_\varepsilon^2 \) implies a higher level of information asymmetries since in the short run, bidders are uncertain about their values. Large bidders tend to possess a better information, and thus make better biddings in the short run. However, when \( T \) increases, bidders are less concerned about asymmetries, so \( \hat{\sigma}_\varepsilon^2 \) should decrease. Both optimization results from 2008 and 2010 auctions show the decreasing trend in \( \hat{\sigma}_\varepsilon^2 \), meaning lower information asymmetries over time, but patterns for \( \hat{\rho} \) are unclear. However, according to bootstrap sampling results in Appendix B, we do not observe obvious increasing trends, but instead we observe comparable values of \( \hat{\sigma}_\varepsilon^2 \) and \( \hat{\rho} \) across times to maturities. This might suggest that bidders’ signals are in fact more or less on the same level across all times to maturities and the correlation between the value parameters is also equivalent. Unfortunately \( \hat{\beta} \) and \( \hat{\theta} \) do not carry as many meaningful interpretations as the other three variables. Please refer to Appendix B for empirical mean and variance of parameters for all datasets.

### 4.3 Tests of the Revenue Equivalence

Parameter estimates allow for the counterfactual analysis to compare revenues. In the discriminatory auction, revenues are collected by the bid functions and the total revenue is given by

\[
D = R^d(\tilde{s},Q) = \int_0^Q p^d(x) \, dx = \frac{1}{c_v} \left( a_vQ + b_v\tilde{s}Q - \frac{1}{2n}Q^2 \right)
\]

In the uniform price auction, the total revenue is given by

\[
U = R^u(\tilde{s},Q) = p^u(Q) Q = \frac{1}{c_u} \left( a_uQ + b_u\tilde{s}Q - \frac{1}{n}Q^2 \right)
\]

where
Note that I compare revenues from the discriminatory auction given by the data and the
simulated best-case Vickrey auction (truthful bidding uniform price auctions), similar to
the methods used in [Hortaçsu and Mcadams(2010)] and [Kastl(2011)]. Then I test the null
hypothesis $H_0$ of equal revenues against the alternative hypothesis $H_1$ that the uniform
price auction generates higher revenues than the discriminatory auction:

$$ H_0 : \frac{1}{N} \sum_{i=1}^{N} (E[U_i] - E[D_i]) = 0 $$

against the one-sided alternative hypothesis

$$ H_1 : \frac{1}{N} \sum_{i=1}^{N} (E[U_i] - E[D_i]) > 0 $$

We can test this hypothesis by using the $t$–statistic

$$ t = \frac{\frac{1}{N} \sum_{i=1}^{N} (E[U_i] - E[D_i])}{\sqrt{\text{Var} \left( \frac{1}{N} \sum_{i=1}^{N} (E[U_i] - E[D_i]) \right)}} $$

The results of the $t$-test are summarized in the following tables. We can compare the
$t$–statistic to the one-sided critical values of $t_{\alpha=0.025, df=N-2} = 2.01$, where $N$ is the number
of auctions in a dataset, for all auctions. Standard errors are calculated in Appendix A.
Conclusions of the test are in the following tables:
In principle, the uniform price auction should generate higher revenues than the discriminatory auction because in the uniform price auction bidders are less vulnerable to the winner's curse and will bid more aggressively. Uniform price auction encourages more competition than the discriminatory auction. Moreover, the uniform price auction might attract more bidders with less access to information to participate since the price everyone pays is the price that incorporates the information held by their better-informed counterparts.

According to the $t$–statistics of the revenue comparison results, we observe no obvious trends in the chance of rejecting or failing to reject the null hypothesis of equal revenues. However, for 2010 T-Bill, we observe a significant drop in $t$–statistic, which changes the conclusion from rejecting the null hypothesis of equal revenues in 28-day and 91-day auctions to failing to reject the null hypothesis in the 182-day auction. This can be explained as follows. Since $T$ increases, common values play more role in determining the bidding behavior of the auctioneers because they are more uncertain of the outcome, resulting in the bids being less aggressive and the participants being less willing to participate. This would no longer allow the uniform price auctions to outperform the discriminatory auctions, thus we fail to reject the null hypothesis of equal revenues between the two auction formats.

Also, observe that for the Treasury auction in 2008 with maturity 28 days, we also fail to reject the null hypothesis of equal revenues. This suggests that the bidding behavior is consistent for the two auction formats, which agree with our finding from the earlier section that this auction dataset is dominated by the private values component so the bidders have bid according to their own valuations, thus outcomes under two auction formats should
yield comparable revenues. This might explain why we fail to reject the null hypothesis in this particular case.

Overall, I believe that the uniform price auction outperforms the discriminatory auction in terms of revenues, especially for the auctions with less time to maturities. For those with longer time to maturities, the two auction mechanism might yield comparable revenue results. However, the revenues generated in the uniform price auction might not be greater than the revenues generated in the discriminatory auction in the market with small number of bidders because of the possibility of the collusion, so the Bank of Thailand should perform the experiment in practice. We provide an indicative test for the existence of the collusion in the next section.

4.4 Tests of Collusion

Given that the Treasury bills market in Thailand consists of less than 20 participants, it is important to incorporate the possibility of collusion into revenue comparison analysis. Suggested by [Athey et al.(2011)Athey, Levin, and Seira], an indicative test for the existence of collusion among bidders is to test for revenue equivalence between the two separate groups. One group consists of auctions with a high number of bidders \( n \geq 11 \), and the other consists of those with low number of bidders \( n < 11 \). We employ a similar test as in the previous section:

\[
H_0 : \frac{1}{N} \sum_{i=1}^{N} (E[U_i] - E[D_i]) = 0
\]

against

\[
H_1 : \frac{1}{N} \sum_{i=1}^{N} (E[U_i] - E[D_i]) > 0
\]

Note that \( N \) represents the number of auctions in each dataset. Results of this one-sided hypothesis testing are included in the following tables.

<table>
<thead>
<tr>
<th>Maturity ((T))</th>
<th>(t)- Statistic</th>
<th>Conclusion at (\alpha = 0.025)</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 days</td>
<td>-1.8232</td>
<td>Fail to reject (H_0)</td>
</tr>
<tr>
<td>91 days</td>
<td>1.6250</td>
<td>Fail to reject (H_0)</td>
</tr>
<tr>
<td>182 days</td>
<td>-0.6254</td>
<td>Fail to reject (H_0)</td>
</tr>
</tbody>
</table>

Table 4.11: Auctions with small number of bidders, 2008 T-Bill
With fewer number of participants, collusion is more likely to exist. Collusion drives the price downward because bidders share their bidding decisions to take advantage of the seller. With collusion, the revenue decreases because of both the decrease in price and the fact that collusion crowds out participation. Moreover, collusion introduces inefficiencies in bidding behavior because of the possibility of more bid shading. In the uniform price auction, collusion decreases the price because bidders share their bidding behaviors to take advantage of the seller. With collusion, the revenue decreases because of both the decrease in price and the fact that collusion crowds out participation, especially in cases where the market has few participants. Thus, if collusion exists, the uniform price auction will not generate significantly higher revenue than the discriminatory auction for markets with few participants. According to the empirical results, it is in general harder to reject the null hypothesis of equal revenue with fewer participants in both 2008 and 2010 auctions. These results indicate the presence of collusion in the market, because bidders no longer bid aggressively under counterfactual uniform price auction mechanism with few people in

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>t− Statistic</th>
<th>Conclusion at α = 0.025</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 days</td>
<td>1.5384</td>
<td>Fail to reject H₀</td>
</tr>
<tr>
<td>91 days</td>
<td>19.9276</td>
<td>Reject H₀</td>
</tr>
<tr>
<td>182 days</td>
<td>24.3109</td>
<td>Reject H₀</td>
</tr>
</tbody>
</table>

Table 4.12: Auctions with large number of bidders, 2008 T-Bill

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>t− Statistic</th>
<th>Conclusion at α = 0.025</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 days</td>
<td>0.2032</td>
<td>Fail to reject H₀</td>
</tr>
<tr>
<td>91 days</td>
<td>0.1780</td>
<td>Fail to reject H₀</td>
</tr>
<tr>
<td>182 days</td>
<td>1.0978</td>
<td>Fail to reject H₀</td>
</tr>
</tbody>
</table>

Table 4.13: Auctions with small number of bidders, 2010 T-Bill

<table>
<thead>
<tr>
<th>Maturity (T)</th>
<th>t− Statistic</th>
<th>Conclusion at α = 0.025</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 days</td>
<td>1.7553</td>
<td>Fail to reject H₀</td>
</tr>
<tr>
<td>91 days</td>
<td>2.1936</td>
<td>Reject H₀</td>
</tr>
<tr>
<td>182 days</td>
<td>10.5050</td>
<td>Reject H₀</td>
</tr>
</tbody>
</table>

Table 4.14: Auctions with large number of bidders, 2010 T-Bill
the market but instead they shade their bids downward. Thus both auction mechanisms generate comparable revenues as a result. For auctions with larger number of participants, we obtain similar findings that the uniform price auctions tend to generate higher revenues than the discriminatory auctions.

For the auctions in 2008 with $T = 28$ days in which the private values effect dominates, we observe that with a large number of bidders, the revenue generated by the uniform price auction is significantly lower than the revenue generated by the discriminatory auction. This suggests that for private values auction with large number of participants and with low likelihood of collusion, the discriminatory auction generates higher revenue than the uniform price auction.

Similar to the discriminatory auction, the uniform price auction may also lead to some inefficiencies. When bidders have some market power, uniform pricing rule motivates bidders to shade bids. However, the convergence to efficiency is rapid as the number of bidders grows, so it is likely that uniform price auction reaches efficient outcomes if the auction is relatively competitive. This hypothesis explains the failure of the uniform price auction to outperform the discriminatory auction in terms of revenues generated by auctions in the environment with large number of auctioneers. Without collusion, in a private values environment, the Treasury auction tends to favor the discriminatory auction.
Chapter 5

Conclusion and Policy Implications

This Honors Thesis has performed an empirical analysis of the Thai Treasury bill auctions in 2008 and 2010 based on the structural econometric model of [Ollikka(2011)] assuming the solution concept of Linear Bayesian demand function equilibria (LBDFE). I estimate the model primitives and bidder valuations and simulate counterfactual auction outcomes from estimated parameter by solving an optimization problem in order to calibrate data with the close-form equilibrium assumption. The optimization involves the use of coordinate descent with grid search. Although it is difficult to make exact interpretations from parameter estimates, we observe that $\hat{\sigma}_\theta^2$ increases with maturities, suggesting that bidders are more uncertain in estimating their own valuation as the time frame increases. Moreover, we observe that for 28-day Treasury auction in 2010, the private values component dominates since $\hat{\rho} = 0$. $\hat{\sigma}_\epsilon^2$ should decrease as time to maturities increase because bidders should be less concerned about information asymmetry over time, while $\hat{\rho}$ should increase as time to maturities increase because of the increasing impact of common values component. However, the trends for $\hat{\sigma}_\epsilon^2$ and $\hat{\rho}$ are not obvious. This might suggest that bidders’ signals are in fact on par across all times to maturities and the correlation between the value parameters is equivalent.

Next, I compare revenue performances of the discriminatory auction and the simulated counterfactual best-case Vickrey auction (truthful bidding uniform price auction). Uniform price auctions should generate higher revenues than discriminatory auctions because in uniform price auctions bidders are less vulnerable to the winner’s curse and bid more aggressively and more effectively. Also uniform price auctions encourage more competition and they attract more bidders with less access to information to participate since the price everyone
pays is the price that incorporates the information held by their better-informed counterparts. Overall results suggest that the uniform price auctions outperform the discriminatory auctions, especially those with less time to maturities. But as time to maturities increase, two mechanisms generate equivalent revenue results because of increasing winner’s curse effects. These results agree with the literatures which agree upon the hypothesis that uniform price auctions generally lead to higher revenues and lead to more efficient bid distributions. However, according to the methodology suggested by [Athey et al.(2011)Athey, Levin, and Seira], I also find that uniform price auctions in a small market such as Thailand are subject to collusion. Collusion drives the price downward because bidders share their bidding decisions to take advantage of the seller. With collusion, the revenue decreases because of both the decrease in price and the fact that collusion crowds out participation. Moreover, collusion introduces inefficiencies in bidding behavior because of the possibility of more bid shading. Results suggest that there is a risk for collusion in the small market which results in equal revenues from both auction formats with fewer participants. But for auctions with larger number of auctioneers, we can still reject the null hypothesis and conclude that the uniform price auctions outperform the discriminatory ones. Without collusion, in a private values environment such as 28-day Treasury auction in 2008, the discriminatory auctions are preferable. Therefore, it is advisable for the Bank of Thailand to experiment issuing uniform price Treasury auctions in order to properly evaluate opposing effects of aggressive bidding and of the winner’s curse effects from this auction mechanism.

The regression-based approach suggested in this Honors Thesis allow us to quickly identify the valuation type implied by the data. Regressing price on number of bidders and regressing standard deviation of price on number of bidders indicate negative slopes on the number of bidder parameter, suggesting that the market is not competitive and bidders are more afraid of the winner’s curse as the number of bidders increase. Regressing coefficient of variation of price on number of bidders also yields negative slopes, suggesting that as the number of bidders increase, the variation in price reduces faster than the level of price, suggesting the strong effect of winner’s curse.

Methodologies suggested in this Honors Thesis and models easily extend to the analysis of any divisible-good auctions. There are no limitations or specifications given by [Ollikka(2011)] or by particular parts of methodologies that are specific to the analysis of Treasury auctions.

Although the models proposed by [Ollikka(2011)] provide the closed-form solution to
the LBDFE, we must acknowledge that this is an approximate equilibrium, which might cause numerical difficulties. However, to the best of my knowledge, this is one of the most recently developed models for the discriminatory and uniform price auctions in the literature of multi-unit auctions.

One way to combat collusion is to set an appropriate level of reserve price. Although the reserve price exists in this data set, I have not included this factor into my revenue comparisons because the Bank of Thailand sets a lenient level of reserve price and accepts almost all bids. However, setting an effective reserve price can be particularly powerful in combating underpricing that might be caused by discriminatory auctions. If the reserve price is triggered, the reserve price becomes the auction clearing price, and only bids at or above that level are accepted. If the number of bids submitted by bidders is usually stable regardless of the level of the reserve price, removing the reserve price will make the price points even more sparse when auctioneer aggregates the projected demand. In that case, the downward bias of the step function bids on the clearing price will be larger due to the wider steps. So setting the proper reserve price can help reduce the underpricing in auctions.

The reserve price is especially important where the bidders have very asymmetric willingness to pay for an item or asymmetric information when participation in the auction is low, both of which are very likely possibilities in our data set. Reserve prices are also very important in reducing the potential damage from collusion because they reduce the profitability of collusion. This is true whether the collusion is tacit or explicit. The importance of reserve prices in limiting collusion is supported by both the theoretical literature and empirical examinations of auction performance such as in [Ausubel and Cranton(2002)]. However, there is not yet a closed form solution of optimal reserve price since there might be multiple Bayesian Nash equilibria of auction outcomes. One way to set an appropriate reserve price is to set it at a level close to, but below, the expected clearing price for the auction, which is likely to be very close to the current price for allowances in the secondary market. Another way to combat collusion is to modify pricing rules. For example, [Conley and Decarolis(2011)] argue that average winning bid rule can help prevent collusions.

Note that the conclusions of this paper may not be immediately applicable to the case of Treasury markets in other countries. Although we have the advantage of using bid-level data, we only observe winning bids and the market is itself rather small with less
than 18 bidders per each auction. Moreover, the Thai Treasury auctions do not have the noncompetitive bidding component. Also, this work has not considered the possibility of resales. With resales, some bidders will not bid aggressively since they can buy the bills in the secondary market.

There are several possible extensions for this paper. We can construct the modified pricing rules to issue the Treasury bill auctions, or to sell objects in divisible-good auctions in general. One possibility is to consider alternative payment mechanisms proposed in [Arman-tier and Sbaï(2009)]. Also, [Ollikka(2011)] assumes Gaussian distribution as a prior for the distributions of signal and bid valuation. It would be an interesting Bayesian statistics problem if we can derive the underlying distribution by resampling with replacement from the data and studying its interesting properties, similar to the methodology suggested by [Hortaçsu and Mcadams(2010)]. Multi-unit auction is still very much an active emerging area of research.
Bibliography


[Hong and Shum(2000)] Hong, Han and Matthew Shum (2000), Structural Estimation of Auction Models.


Appendix A

Derivation of the standard error estimates

The standard error of \( \frac{1}{n} \sum_{i=1}^{n} (E[U_i] - E[D_i]) \) is calculated by the following procedure. Let \( \Lambda = (\beta, \bar{\theta}, \sigma_{\bar{\theta}}^2, \sigma_{\epsilon}^2, \rho) \) be a vector of all parameters. Define \( g_i(\Lambda) = E[U_i] - E[D_i] \) for \( i = 1, 2, 3, ..., n \), where \( i \) is the index for different auctions in a data. Let \( H = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2}{\partial \Lambda^2} g_i(\Lambda) \) and \( S = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_i(\Lambda)}{\partial \Lambda} \frac{\partial g_i(\Lambda)}{\partial \Lambda}^T \), then the asymptotic robust standard error is given by

\[
AVar(g(\Lambda)) = \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2}{\partial \Lambda^2} g_i(\Lambda) \right]^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\partial g_i(\Lambda)}{\partial \Lambda} \frac{\partial g_i(\Lambda)}{\partial \Lambda}^T \right) \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2}{\partial \Lambda^2} g_i(\Lambda) \right]^{-1} = H^{-1}SH^{-1}
\]

Since first-order and second-order derivatives are analytically intractable, we can approximate them by using the finite difference method as follows:

\[
\frac{\partial f}{\partial x} = \lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x - \epsilon)}{2\epsilon}
\]

\[
\frac{\partial^2 f}{\partial x^2} = \lim_{\epsilon \to 0} \frac{f'(x + \epsilon) - f'(x - \epsilon)}{2\epsilon} = \lim_{\epsilon \to 0} \frac{f(x + 2\epsilon) - f(x - 2\epsilon) - 2f(x)}{4\epsilon^2}
\]

where \( \epsilon > 0 \) is a small positive number. In case of computing the elements of mixed partial derivative for elements in \( H^{-1} \), we define \( e_1 \) and \( e_2 \) to be canonical basis. Let \( \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \) be a vector of parameters. Then we can approximate the mixed partial derivative at a point \( x \) as follows:
\[
\left. \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2} \right|_{x} = \frac{\partial}{\partial \theta_1} \frac{\partial f}{\partial \theta_2} \\
= \frac{\partial f(x + \epsilon e_2) - f(x - \epsilon e_2)}{2\epsilon} \\
= \frac{1}{2\epsilon} \left[ \frac{\partial}{\partial \theta_1} f(x + \epsilon e_2) - \frac{\partial}{\partial \theta_1} f(x - \epsilon e_2) \right] \\
= \frac{1}{4\epsilon^2} \left[ f(x + \epsilon e_2 + \epsilon e_1) - f(x + \epsilon e_2 - \epsilon e_1) - f(x - \epsilon e_2 + \epsilon e_1) + f(x - \epsilon e_2 - \epsilon e_1) \right]
\]

Note that in approximating derivatives by finite differences, we need to be careful in making sure that the parameters do not leave the domain. For example, we have the constraints that \( \hat{\sigma}_\theta^2 \) and \( \hat{\sigma}_\epsilon^2 \) must be positive or 0 and \( 0 \leq \hat{\rho} \leq 1 \). I already incorporate these constraints in the MATLAB implementation.

Let \( \Lambda_0 \) be the vector of estimated parameters (from the optimization). The asymptotic distribution of \( g(\Lambda) = \frac{1}{n} \sum_{i=1}^{n} g_i(\Lambda) \) is given by

\[
\sqrt{n} (g(\Lambda) - g(\Lambda_0)) \xrightarrow{D} N \left( 0, \nabla g(\Lambda_0)^T \text{Var}(g(\Lambda_0)) \nabla g(\Lambda_0) \right)
\]

This yields the asymptotic standard error of \( g(\Lambda) \)

\[
s.e. (g(\Lambda_0)) = \sqrt{\frac{\nabla g(\Lambda_0)^T \text{Var}(g(\Lambda_0)) \nabla g(\Lambda_0)}{n}}
\]
Appendix B

Empirical Distributions

We understand the empirical distributions of parameter estimates by using the bootstrap resampling similar to the mechanism originated by [Efron and Tibshirani(1979)]: in each year for each type of maturity (one dataset), take auction outcomes (including all winning price-quantity pairs up to the quota limit) as data points, then resample all outcomes with replacement (make random draws from the empirical distribution). Then follow similar strategies above to derive model estimates and compare revenue counterfactuals. These distributions allow us to calculate the parameters’ means and variances, along with their 95% confidence intervals.
Table B.1: Empirical Distributions of Parameters, 2008 T-Bill, $T = 28$ days

Please note that the histograms of empirical distributions of parameters for the other datasets are omitted. Here are the empirical mean and variance of parameters for all datasets.
Table B.2: Empirical Mean and Variance of Parameters, 2008 T-Bill, $T = 28$ days

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Mean</th>
<th>Empirical Variance</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.1983</td>
<td>0.0016</td>
<td>[0.0965, 0.2458]</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-12.5509</td>
<td>615.3092</td>
<td>[-72.4268, 43.4816]</td>
</tr>
<tr>
<td>$\sigma^2_0$</td>
<td>8.1241</td>
<td>80.6815</td>
<td>[0, 20.1646]</td>
</tr>
<tr>
<td>$\sigma^2_e$</td>
<td>0.5214</td>
<td>0.1086</td>
<td>[0, 1.0485]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0874</td>
<td>0.0413</td>
<td>[0, 0.8577]</td>
</tr>
</tbody>
</table>

$N = 244$ Bootstrap Samples

Table B.3: Empirical Mean and Variance of Parameters, 2008 T-Bill, $T = 91$ days

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Mean</th>
<th>Empirical Variance</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.0285</td>
<td>0.0043</td>
<td>[0.0743, 0.2454]</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-12.3558</td>
<td>1,154.7</td>
<td>[-72.7493, 46.4185]</td>
</tr>
<tr>
<td>$\sigma^2_0$</td>
<td>8.0181</td>
<td>85.5690</td>
<td>[0, 20.6906]</td>
</tr>
<tr>
<td>$\sigma^2_e$</td>
<td>0.5007</td>
<td>0.0966</td>
<td>[0, 1.0981]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0796</td>
<td>0.0321</td>
<td>[0, 0.8661]</td>
</tr>
</tbody>
</table>

$N = 219$ Bootstrap Samples

Table B.4: Empirical Mean and Variance of Parameters, 2008 T-Bill, $T = 182$ days

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Mean</th>
<th>Empirical Variance</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.1518</td>
<td>0.0293</td>
<td>[0.0316, 0.4933]</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-20.5482</td>
<td>1,172.6</td>
<td>[-85.0589, 41.0348]</td>
</tr>
<tr>
<td>$\sigma^2_0$</td>
<td>8.6838</td>
<td>91.2922</td>
<td>[0, 31.3816]</td>
</tr>
<tr>
<td>$\sigma^2_e$</td>
<td>0.5544</td>
<td>0.0883</td>
<td>[0, 0.9060]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0763</td>
<td>0.0322</td>
<td>[0, 0.8590]</td>
</tr>
</tbody>
</table>

$N = 243$ Bootstrap Samples
### APPENDIX B. EMPIRICAL DISTRIBUTIONS

#### Table B.5: Empirical Mean and Variance of Parameters, 2010 T-Bill, $T = 28$ days

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Mean</th>
<th>Empirical Variance</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.1114</td>
<td>0.0100</td>
<td>[0.0382, 0.2963]</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$-8.1258$</td>
<td>990.9138</td>
<td>$[-63.4395, 70.0316]$</td>
</tr>
<tr>
<td>$\sigma_{\theta}^2$</td>
<td>6.9588</td>
<td>112.9975</td>
<td>[0, 31.0903]</td>
</tr>
<tr>
<td>$\sigma_{\epsilon}^2$</td>
<td>0.5478</td>
<td>0.6970</td>
<td>[0, 0.9514]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0898</td>
<td>0.0255</td>
<td>[0, 0.5657]</td>
</tr>
</tbody>
</table>

$N = 225$ Bootstrap Samples

#### Table B.6: Empirical Mean and Variance of Parameters, 2010 T-Bill, $T = 91$ days

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Mean</th>
<th>Empirical Variance</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.1114</td>
<td>0.0205</td>
<td>[0.0204, 0.4524]</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$-14.4042$</td>
<td>555.1417</td>
<td>$[-68.3059, 26.7464]$</td>
</tr>
<tr>
<td>$\sigma_{\theta}^2$</td>
<td>8.4606</td>
<td>70.5727</td>
<td>[0, 22.9127]</td>
</tr>
<tr>
<td>$\sigma_{\epsilon}^2$</td>
<td>0.5714</td>
<td>0.0803</td>
<td>[0, 0.9017]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0446</td>
<td>0.0120</td>
<td>[0, 0.3436]</td>
</tr>
</tbody>
</table>

$N = 220$ Bootstrap Samples

#### Table B.7: Empirical Mean and Variance of Parameters, 2010 T-Bill, $T = 182$ days

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Mean</th>
<th>Empirical Variance</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0142</td>
<td>[0.0597, 0.4478]</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$-19.1623$</td>
<td>805.3398</td>
<td>$[-79.2142, 28.7314]$</td>
</tr>
<tr>
<td>$\sigma_{\theta}^2$</td>
<td>9.8702</td>
<td>187.0718</td>
<td>[0, 32.2914]</td>
</tr>
<tr>
<td>$\sigma_{\epsilon}^2$</td>
<td>0.4252</td>
<td>0.1102</td>
<td>[0, 0.9374]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0702</td>
<td>0.0237</td>
<td>[0, 0.5720]</td>
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$N = 228$ Bootstrap Samples