Two-way fixed effects estimators with heterogeneous treatment effects

Clément de Chaisemartin† Xavier D’Haultfoeuvre‡

March 30, 2018

Abstract

Around 20% of all empirical articles published by the American Economic Review between 2010 and 2012 estimate treatment effects using linear regressions with time and group fixed effects. In a model where the effect of the treatment is constant across groups and over time, such regressions identify the treatment effect of interest under the standard “common trends” assumption. But these regressions have not been analyzed yet allowing for treatment effect heterogeneity. We show that under two alternative sets of assumptions, such regressions identify weighted sums of average treatment effects in each group and period, where some weights may be negative. The weights can be estimated, and can help researchers assess whether their results are robust to heterogeneous treatment effects across groups and periods. When many weights are negative, their estimates may not even have the same sign as the true average treatment effect if treatment effects are heterogenous. We also propose another estimator of the treatment effect that does not rely on any homogeneity assumption. Finally, we estimate the weights in two applications and find that in both cases, around half of the average treatment effects receive a negative weight.

1 Introduction

A popular method to identify treatment effects in the absence of experimental data is to compare groups experiencing different evolutions of their exposure to the treatment over time. Under the assumption that their outcome would have followed the same evolution if their exposure had also followed the same evolution, any difference we observe in the evolution of the outcome between those groups should be due to the treatment effect. In practice, this idea is implemented by

*We are very grateful to Jesse Shapiro and seminar participants at Bergen, CREST, and UCSB for their helpful comments.
†University of California at Santa Barbara, clementdechaisemartin@ucsb.edu
‡CREST, xavier.dhaultfoeuvre@ensae.fr

1
estimating linear regressions controlling for both group and time fixed effects, which we hereafter refer to as two-way fixed effects regressions. We conducted a literature review, and found that 19.6% of all empirical articles published by the American Economic Review between 2010 and 2012 use two-way fixed effects regressions. When the effect of the treatment is constant across groups and over time, such regressions identify the treatment effect of interest under the standard “common trends” assumption. While assuming constant treatment effects across groups and over time may sometimes not be plausible, those regressions have not been analyzed yet without that assumption. This is the purpose of this paper.

We study the coefficient of the treatment variable in three popular two-way fixed effects regressions. We consider two alternative sets of assumptions. The first comprises solely the standard common trends assumption. The second also includes an assumption requiring that some average treatment effects remain stable over time, and another assumption requiring that the treatment follow a monotonic evolution within each group and between each pair of consecutive time periods. We start by showing that under the first set of assumptions, the coefficient of the treatment variable in those three regressions identifies a weighted sum of the average treatment effect on treated units (ATT) in each group and at each period, with weights that differ across regressions. In most applications, some of those weights are negative. Indeed, for each regression, we derive the condition that must hold to have that all weights are positive, and we show that it is very restrictive. We then show that under the second set of assumptions, the coefficient of the treatment variable in those three regressions identifies a weighted sum of the local average treatment effect (LATE) of the “switchers” in each group and at each period, where “switchers” refers to units that experience a change in their treatment between two periods. Here as well, the weights differ across regressions. We show that the weights attached to the second regression we consider, the first-difference regression, may sometimes all be positive. On the other hand, in most instances some of the weights attached to the other two regressions are negative.

Importantly, the weights attached to each coefficient under each set of assumptions can be estimated. Estimating those weights may help researchers assess whether the causal interpretation of those coefficients is robust to heterogeneous treatment effects across groups and periods, and it may help them choose which regression to estimate.¹ When many of the weights attached to a coefficient are negative, it does not satisfy the no-sign-reversal property if the treatment effect is heterogeneous across groups and periods: even if the treatment effect is positive for all units in the population, this coefficient may be negative. We also show that a simple function of the weights identifies the minimal value of the standard deviation of the treatment effect across groups and periods under which the overall average treatment effect may actually be of the opposite sign than that of the regression coefficient. The larger this lower bound, the more robust the coefficient is to heterogenous treatment effects. Comparing the proportion of negative

¹Stata programs performing those estimations can be found on the authors’ websites.
weights and the lower bound on the standard deviation of treatment effects attached to each coefficient may be a way to choose which regression to estimate.

We derive our main results with a binary treatment and without covariates in the regressions. However, we show that our results can be extended to ordered treatments and to regressions with covariates. Interestingly, the weights remain unchanged in these two cases. Therefore, our results are widely applicable in practice. We also propose an estimator of the LATE of switchers in all groups and time periods that does not rely on any homogeneous treatment effect assumption. That estimator can be used when for each pair of consecutive dates, there are groups whose exposure to the treatment does not change between those two dates, a condition that is often met in applications.2

Finally, we review two articles that have used the regressions we study, and estimate the weights attached to those regressions. In both cases, more than half of the weights are negative under the common trends assumption alone. Under our second set of assumptions, the first article still has more than half of negative weights, but the second article has only positive weights. Therefore, the coefficient in the second article is robust to heterogeneous switchers’ LATEs across groups and periods, but the coefficient in the first article is not. Our second set of assumptions still requires that some average treatment effects be stable over time. Our new estimator does not rely on any homogeneous treatment effect assumption. We estimate it in the second article, and we find that it is significantly larger than the coefficient estimated by the authors.

Our paper is related to two strands of the literature. First, it is related to papers studying the consequences of treatment effect heterogeneity in linear models. For instance, White (1980) shows that in a model with heterogeneous treatment effects, ordinary least square estimators identify a weighted average of marginal effects, and he exhibits the corresponding weights. An important difference between those and our paper is that in our setting, treatment effect heterogeneity may entail negative weights. Our paper is also related to the literature on differences-in-differences. In particular, the regression coefficients we study are generalizations of the so-called Wald-DID estimand studied by de Chaisemartin and D’Haultfoeuille (2017a). Indeed, this estimand is equal to the three coefficients we consider when the data only bears two groups and two periods. Our main result may thus be seen as an extension of their Theorem 1 to multiple periods and groups. In fact, a preliminary version of our main result appeared in a working paper version of that paper (see Theorems S1 and S2 in de Chaisemartin and D’Haultfoeuille, 2015). In research independent from ours, Borusyak and Jaravel (2016) study the staggered adoption design, a research design where the first regression we study has been used. They show that if the common trends assumption holds, and if the treatment effect only depends on the number of periods elapsed since a unit has started receiving the treatment, then that regression estimates a weighted sum of the average treatment effect at each period, with weights that may

2A Stata package computing that estimator can be found on the authors’ websites.
be negative. This result can be obtained from our result for that regression under the common trends assumption, by further assuming that the treatment effect only depends on the number of periods elapsed since treatment was received.

The paper is organized as follows. Section 2 presents the three two-way fixed effects regressions we consider, our two sets of assumptions, and our parameters of interest. Section 3 presents our main results. Section 4 extends these results to non-binary treatments and regressions with covariates. Section 5 presents our alternative estimand. Section 6 presents our literature review of the articles published in the AER, and the results of our two empirical applications.

2 Two-way fixed effects regressions, assumptions, and parameters of interest

Assume that one is interested in measuring the effect of a treatment $D$ on some outcome $Y$. We first assume for simplicity that $D$ is binary, but most of our results apply to any ordered treatment, as we show in Subsection 4.1. Then, let $Y(0)$ and $Y(1)$ denote the two potential outcomes of the same unit without and with the treatment. The observed outcome is $Y = Y(D)$. We also assume that the population can be divided into a finite number of time periods represented by a random variable $T \in \{0, 1, \ldots, \tilde{t}\}$, and into a finite number of groups represented by a random variable $G \in \{0, 1, \ldots, \tilde{g}\}$. Each group $\times$ period cell may bear several or a single unit. In the latter case, conditional on $G$ and $T$ all the random variables we consider are degenerate. In the former case, those variables may have a non-degenerate distribution. Some of the assumptions we introduce below have a different interpretation in each of these two cases, as we highlight in the end of this section by reviewing several concrete examples.

We now introduce some notation we use throughout the paper. For any random variable $R$ and for every $(g, t) \in \{0, \ldots, \tilde{g}\} \times \{0, \ldots, \tilde{t}\}$, let $R_{g,\cdot}$, $R_{\cdot,t}$, and $R_{g,t}$ respectively be random variables such that $R_{g,\cdot} \sim R|G = g$, $R_{\cdot,t} \sim R|T = t$, and $R_{g,t} \sim R|G = g, T = t$, where $\sim$ denotes equality in distribution. Finally, let $FD_R(g, t) = E(R_{g,t}) - E(R_{g,t-1})$ denote the conditional first-difference operator. This notational shortcut is useful to avoid the notational burden of, e.g., evaluating the function $(g, t) \mapsto E(D_{g,t})$ at $(G, T - 1)$.

We study the three following linear regressions coefficients.

**Regression 1** Let $\beta_1$ denote the coefficient of $E(D|G, T)$ in an OLS regression of $Y$ on a constant, $(1\{G = g\})_{1 \leq g \leq \tilde{g}}$, $(1\{T = t\})_{1 \leq t \leq \tilde{t}}$, and $E(D|G, T)$.

---

3To draw inference on our parameters of interest, we will let the number of groups go to infinity, but for now we assume that this number is fixed.
Regression 2 Let $\beta_2$ denote the coefficient of $FD_D(G, T)$ in an OLS regression of $FD_Y(G, T)$ on a constant, $(1\{T = t\})_{2 \leq t \leq \bar{t}}$, and $FD_D(G, T)$, conditional on $T \geq 1$.

Regression 3 If $\bar{t} = 1$, let $\beta_3$ denote the coefficient of $D$ in a 2SLS regression of $Y$ on a constant, $(1\{G = g\})_{1 \leq g \leq g}$, $1\{T = 1\}$, and $D$, where the instrument for $D$ is $f(G)1\{T = 1\}$ for some real-valued function $f$ defined on $\{1, \ldots, \bar{g}\}$.

Regression 1 is the OLS regression of the outcome on group and time fixed effects, and on the mean value of the treatment in the group a unit belongs to and at the period she is observed. This regression is very pervasive: 11 articles published in the AER between 2010 and 2012 have estimated it. Other articles have estimated regressions similar to it, e.g. with two treatment variables instead of one.

Regression 2 is the OLS regression of the group-level first-difference of the mean outcome on the group-level first-difference of the mean treatment, with time fixed effects. It is also very pervasive: seven articles published in the AER between 2010 and 2012 have estimated it, and other articles have estimated regressions similar to it. When $\bar{t} = 1$ and $T \perp \perp G$, meaning that groups’ distribution remains stable over time, one can show using the Frisch-Waugh theorem that $\beta_1 = \beta_2$. $\beta_1$ usually differs from $\beta_2$ when $\bar{t} > 1$ or $T$ is not independent from $G$.

Regression 3 is a 2SLS regression with group fixed effects and a time indicator, using the interaction of a group-level variable and the time indicator to instrument for the treatment. $f(G)$ typically represents the intensity of an incentive to get treated administered at the group level, while $T = 0$ and $T = 1$ typically represent the periods before and after the incentive is administered. For instance, in Duflo (2001), $f(G)$ is the number of schools constructed in each Indonesian district during a primary school construction program, $T = 0$ (resp. $T = 1$) for cohorts that completed primary school before (resp. after) the program, and $D$ is years of schooling completed. Regression 3 is quite pervasive: three articles published in the AER between 2010 and 2012 have estimated it, and other articles have estimated regressions similar to it. It is related to Regression 1. When $\bar{t} = 1$, $\beta_1$ can be estimated by a 2SLS regression of $Y$ on a constant, $(1\{G = g\})_{1 \leq g \leq g}$, $1\{T = 1\}$, and $D$, where the instruments for $D$ are all the interactions between the groups and the period 1 indicators. Regression 3 uses only one instrument, namely $f(G)1\{T = 1\}$, instead of all the interactions $1\{G = g\}1\{T = 1\}$ for $g \in \{1, \ldots, \bar{g}\}$. Therefore, if $\bar{g} = 1$, $\beta_1 = \beta_3$. When $\bar{g} > 1$, $\beta_1$ usually differs from $\beta_3$.

We now introduce the main assumptions we consider.

Assumption 1 (Common trends) For all $t \in \{1, \ldots, \bar{t}\}$, $E(Y(0)|G, T = t) - E(Y(0)|G, T = t - 1)$ does not depend on $G$.

Assumption 2 (Monotonicity of the treatment between consecutive periods in each group) There exist random variables $D(0), \ldots, D(\bar{t})$ such that:
1. \( D = D(T) \);

2. For all \( t \in \{1, \ldots, \bar{t}\} \), \( D(t) \perp T \mid G \);

3. For all \( t \in \{1, \ldots, \bar{t}\} \), \( P(D(t) \geq D(t-1) \mid G) = 1 \) or \( P(D(t) \leq D(t-1) \mid G) = 1 \).

**Assumption 3** (Homogeneous ATTs across periods) For all \((g, t) \in \{0, \ldots, g\} \times \{1,\ldots, \bar{t}\} \),
\[
E(Y(1) - Y(0) \mid G = g, T = t, D(t-1) = 1) = E(Y(1) - Y(0) \mid G = g, T = t - 1, D(t-1) = 1).
\]

Assumption 1 requires that the mean of \( Y(0) \) follow the same evolution over time in the treatment and control groups. This assumption also underlies the standard DID estimand (see, e.g., Abadie, 2005). Assumption 2 requires that each unit has \( \bar{t} + 1 \) variables \( D(0), D(1), \ldots, D(\bar{t}) \) attached to her, which respectively denote her treatment at \( T = 0, 1, \ldots, \bar{t} \). It also requires that the distribution of those variables be stable across periods in each group. Finally, it implies that between each pair of consecutive periods, in a given group there cannot be both units whose treatment increases and units whose treatment decreases. Assumption 3 requires that in every group, the average treatment effect among units treated in period \( t-1 \) does not change between \( t-1 \) and \( t \). This assumption also underlies the standard Wald-DID estimand (see de Chaisemartin and D’Haultfoeuille, 2017a).

We now review several examples to clarify the restrictions implied by Assumption 2. Some articles have estimated regression 1, 2, or 3 with a treatment that is constant within each group × period cell. For instance, in Gentzkow et al. (2011) the treatment is the number of newspapers in county \( g \) and election year \( t \). Then, point 3 of Assumption 2 automatically holds: \( P(D(t) \geq D(t-1) \mid G) = 1 \) in groups where the treatment increases between \( t-1 \) and \( t \), and \( P(D(t) \leq D(t-1) \mid G) = 1 \) in groups where it decreases. Point 2 of Assumption 2 will also automatically hold if the regression is estimated at the group × period-level, unless some groups appear or disappear over time. Other articles have estimated regression 1, 2, or 3 with a treatment that varies within each group × period cell. For instance, in Enikolopov et al. (2011), the treatment is whether someone has access to an independent TV channel, groups are Russia’s regions, and periods are election years. Then, point 3 of Assumption 2 will fail to hold if there are groups where the treatment of some units diminishes between \( t-1 \) and \( t \), while the treatment of other units increases. When regression 1, 2, or 3 is estimated with individual-level panel data, \( D(0), D(1), \ldots, \) and \( D(\bar{t}) \) are observed: those variables are just the treatments of each unit at each period. Then, one can assess from the data whether point 3 of Assumption 2 holds or not.\(^4\) On the other hand, when regression 1, 2, or 3 is estimated with individual-level repeated cross-sections, only \( D(T) \) is observed so point 3 of Assumption 2 is not testable.

\(^4\)This test may reveal that point 3 of Assumption 2 fails. However, our results still hold if the treatment variables satisfy the threshold crossing model in Equation (3.2) in de Chaisemartin and D’Haultfoeuille (2017a), which is weaker than point 3 of Assumption 2.
In this paper, our two parameters of interest are $\Delta^{TR} = E(Y(1) - Y(0)|D = 1)$ and $\Delta^{S} = E(Y(1) - Y(0)|S)$, where $S = \{D(T - 1) \neq D(T), T \geq 1\}$ denotes units whose treatment status switches between $T - 1$ and $T$. $\Delta^{TR}$ is the Average Treatment effect on the Treated (ATT), and $\Delta^{S}$ is the Local Average Treatment Effect (LATE) of the switchers.

3 Main results

3.1 Two-way fixed effects estimators as weighted sums of average treatment effects

For any $(g, t) \in \{0, ..., \bar{g}\} \times \{0, ..., \bar{t}\}$, let $\Delta^{TR}_{g,t} = E(Y(1) - Y(0)|D = 1, G = g, T = t)$ denote the ATT in group $g$ and at period $t$. For any $(g, t) \in \{0, ..., \bar{g}\} \times \{1, ..., \bar{t}\}$, let $\Delta^{S}_{g,t} = E(Y(1) - Y(0)|S, G = g, T = t)$ denote the LATE of switchers in group $g$ and at period $t$. We have $\Delta^{TR} = E[\Delta^{TR}_{G,T}|D = 1]$ and $\Delta^{S} = E[\Delta^{S}_{G,T}|S]$: $\Delta^{TR}$ and $\Delta^{S}$ are respectively equal to weighted averages of the $(\Delta^{TR}_{g,t})_{g,t}$ and $(\Delta^{S}_{g,t})_{g,t}$. We now show that under Assumption 1 (resp. Assumptions 1-3) above, $\beta_1, \beta_2, \beta_3$ identify weighted sums of $(\Delta^{TR}_{g,t})_{g,t}$ (resp. $(\Delta^{S}_{g,t})_{g,t}$), with weights that can potentially be negative.

We start by introducing defining the weights attached to $(\Delta^{TR}_{g,t})_{g,t}$ in Regressions 1, 2, and 3 under Assumption 1. First, for any $(g, t) \in \{0, ..., \bar{g}\} \times \{0, ..., \bar{t}\}$, let $\varepsilon_{1,g,t}$ denote the residual of observations in group $g$ and at period $t$ in the regression of $E(D|G, T)$ on a constant, $(1\{G = g\})_{1 \leq g \leq \bar{g}}$, and $(1\{T = t\})_{1 \leq t \leq \bar{t}}$. Then, provided that $E[\varepsilon_{1,G,T}E(D|G, T)] \neq 0$, let

$$w_{1,g,t} = \frac{\varepsilon_{1,g,t}E(D)}{E[\varepsilon_{1,G,T}E(D|G, T)]}, \quad W_1 = w_{1,G,T}.$$ 

As in this section we assume that the treatment is binary, we have $W_1 = \varepsilon_{1,G,T}/E(\varepsilon_{1,G,T}|D = 1)$. However, this simplification does not carry through to the case where the treatment is not binary, so we stick to the definition of $W_1$ in the previous display to ensure it applies both to the binary- and non-binary-treatment cases.

Second, for any $(g, t) \in \{0, ..., \bar{g}\} \times \{1, ..., \bar{t}\}$, let $\varepsilon_{2,g,t}$ denote the residual of observations in group $g$ and at period $t$ in the regression of $FD_D(G, T)$ on a constant and $(1\{T = t\})_{2 \leq t \leq \bar{t}}$, conditional on $T \geq 1$. For any $g \in \{0, ..., \bar{g}\}$, let also $\varepsilon_{2,g,0} = 0$, $\varepsilon_{2,g,1} = 0$. Then, for any $(g, t) \in \{0, ..., \bar{g}\} \times \{0, ..., \bar{t}\}$, let

$$\tilde{w}_{2,g,t} = \varepsilon_{2,g,t} - \varepsilon_{2,g,t+1} \frac{P(G = g, T = t + 1)}{P(G = g, T = t)}.$$ 

Finally, provided that $E[\tilde{w}_{2,G,T}E(D|G, T)] \neq 0$, let

$$w_{2,g,t} = \frac{\tilde{w}_{2,g,t}E(D)}{E[\tilde{w}_{2,G,T}E(D|G, T)]}, \quad W_2 = w_{2,G,T}.$$
Third, for any \((g, t) \in \{0, ..., \bar{g}\} \times \{1, 0\}\), let \(\varepsilon_{3,g,t}\) denote the residual of observations in group \(g\) and at period \(t\) in the regression of \(f(G)1\{T = 1\}\) on a constant, \((1\{G = g\})_{1 \leq g \leq \bar{g}}\) and \(1\{T = 1\}\). Provided that \(E[\varepsilon_{3,G,T}E(D|G, T)] \neq 0\), let

\[
w_{3,g,t} = \frac{\varepsilon_{3,g,t}E(D)}{E[\varepsilon_{3,G,T}E(D|G, T)]}, \quad W_3 = w_{3,G,T}.
\]

Note that for every \(k \in \{1, 2, 3\}\), \(E(W_k|D = 1) = 1\): conditional on \(D = 1\), \(W_k\) is a weighting variable with an expectation equal to 1.

We then define the weights attached to \((\Delta^S_{g,t})_{g,t}\) in Regressions 1, 2, and 3 under Assumptions 1-3. First, for all \((g, t) \in \{0, ..., \bar{g}\} \times \{1, ..., \bar{t}\}\), let

\[
s_{g,t} = \text{sgn}\left(E(D_{g,t}) - E(D_{g,t-1})\right),
\]

where for any real number \(x\), \(\text{sgn}(x) = 1\{x > 0\} - 1\{x < 0\}\). \(s_{g,t}\) is equal to 1 (resp. -1, 0) for groups where the share of treated units increases (resp. decreases, does not change) between \(t - 1\) and \(t\). Let also, for all \((g, t) \in \{0, ..., \bar{g}\} \times \{1, ..., \bar{t}\}\),

\[
\tilde{\omega}_{1,g,t} = \frac{s_{g,t}E[\varepsilon_{1,G,T}1\{G = g, T \geq t\}]}{P(G = g, T = t)}.
\]

Finally, provided that \(E[\tilde{\omega}_{1,G,T}|S] \neq 0\), let

\[
\omega_{1,g,t} = \frac{\tilde{\omega}_{1,g,t}}{E[\tilde{\omega}_{1,G,T}|S]}, \quad \Omega_1 = \omega_{1,G,T}.
\]

Second, for \(k = 2, 3\) and provided that \(E[s_{G,T}\varepsilon_{k,G,T}|S] \neq 0\), for all \((g, t) \in \{0, ..., \bar{g}\} \times \{1, ..., \bar{t}\}\), let

\[
\omega_{k,g,t} = \frac{s_{g,t}\varepsilon_{k,g,t}}{E[s_{G,T}\varepsilon_{k,G,T}|S]], \quad \Omega_k = \omega_{k,G,T}.
\]

Note that for every \(k \in \{1, 2, 3\}\), \(E(\Omega_k|S) = 1\): in the population of switchers, \(\Omega_k\) is a weighting variable with an expectation equal to 1.

Importantly, \((w_{k,g,t})_{g,t}\) and \((\omega_{k,g,t})_{g,t}\) are identified and can easily be estimated. This is obvious for \((w_{k,g,t})_{g,t}\), but less so for \((\omega_{k,g,t})_{g,t}\), as their denominators involve expectations over switchers, a population that is generally not identified. Nonetheless, we show in the appendix that under Assumption 2, for any function \(f\), \(E(f(G, T)|S)\) is identified by

\[
E(f(G, T)|S) = \frac{\sum_{(g,t):t \geq 1} P(G = g, T = t)|E(D_{g,t}) - E(D_{g,t-1})| f(g,t)}{\sum_{(g,t):t \geq 1} P(G = g, T = t)|E(D_{g,t}) - E(D_{g,t-1})|}. \quad (1)
\]

We can now state our main result. We say below that \(\beta_k\) (resp. \(W_k, \Omega_k\)) is well defined whenever there is a unique solution to the linear system corresponding to the regression attached to \(\beta_k\) (resp. when the denominator in their definition is not zero).
Theorem 1 Assume that $D$ is binary, $k \in \{1, 2\}$ or $k = 3$ and $T = 1$, and $\beta_k$ is well defined.

1. If Assumption 1 holds, then $W_k$ is well defined and 
   $$
   \beta_k = E \left[ W_k \Delta_{g,T}^{TR} | D = 1 \right].
   $$

2. If Assumptions 1-3 hold, then $\Omega_k$ is well defined and 
   $$
   \beta_k = E \left[ \Omega_k \Delta_{g,T}^{S} | S \right].
   $$

In Theorem 1, we show that under Assumption 1 (resp. Assumptions 1-3), $\beta_1$, $\beta_2$, and $\beta_3$ identify weighted sums of the ATTs (resp. of switchers’ LATEs) in each group and at each period. Indeed, using the law of iterated expectations, the displayed equations in the first and second statements of the theorem are respectively equivalent to

$$
\beta_k = \sum_{g=0}^{G} \sum_{t=0}^{T} P(G = g, T = t | D = 1) w_{k,g,t} \Delta_{g,t}^{TR} \tag{2}
$$

$$
\beta_k = \sum_{g=0}^{G} \sum_{t=1}^{T} P(G = g, T = t | S) \omega_{k,g,t} \Delta_{g,t}^{S}. \tag{3}
$$

Researchers can estimate the weights $(P(G = g, T = t | D = 1) w_{k,g,t})_{g,t}$ and $(P(G = g, T = t | S) \omega_{k,g,t})_{g,t}$ in the application they consider. In particular, they may want to check whether some of the $(P(G = g, T = t | D = 1) w_{k,g,t})_{g,t}$ (resp. $(P(G = g, T = t | S) \omega_{k,g,t})_{g,t}$) are negative. If so, $\beta_k$ does not satisfy the no-sign-reversal property under Assumption 1 (resp. Assumptions 1-3): $\beta_k$ may for instance be negative while the treatment effect is positive for everybody in the population.

Actually, in most instances, some of the weights in Equation (2) will be negative, thus implying that $\beta_k$ does not satisfy the no-sign-reversal property under Assumption 1. First, note that $w_{1,g,t} = \varepsilon_{1,g,t} E(D)/V(\varepsilon_{1,G,T})$. As $E(\varepsilon_{1,G,T}) = 0$, there must be some values of $(g,t)$ for which $\varepsilon_{1,g,t} < 0$, otherwise we would have $\varepsilon_{1,g,t} = 0$ for all $(g,t)$ and $\beta_1$ would not be well defined. Therefore, some of the $(w_{1,g,t})_{g,t}$ must be negative. Accordingly, the only instance where all the weights $(P(G = g, T = t | D = 1) w_{1,g,t})_{g,t}$ are positive is when all the $(g,t)$ such that $w_{1,g,t} < 0$ are also such that $E(D_{g,t}) = 0$, thus implying that $P(G = g, T = t | D = 1) = 0$. This condition is for instance satisfied in sharp DID with two groups and two periods and where $E(D_{g,t}) = 1 \{g = 1, t = 1\}$. But it is unlikely to hold more generally. It requires that any $(g,t)$ such that $E(D_{g,t})$ is lower than its predicted value in a linear regression with group and time fixed effects is also such that $E(D_{g,t}) = 0$. For instance, when $E(D_{g,t}) > 0$ for all $(g,t)$, some of the weights $(P(G = g, T = t | D = 1) w_{1,g,t})_{g,t}$ must be negative. Similarly, one can show that
some of the \((w_{2,g,t})_{g,t}\) and \((w_{3,g,t})_{g,t}\) must be negative.\(^5\) Hence, the only instance where all the weights \((P(G = g, T = t|D = 1)w_{2,g,t})_{g,t}\) (resp. \((P(G = g, T = t|D = 1)w_{3,g,t})_{g,t}\)) are positive is when all the \((g,t)\) such that \(w_{2,g,t} < 0\) (resp. \(w_{3,g,t} < 0\)) are also such that \(E(D_{g,t}) = 0\).

In some instances, the weights in Equation (3) will all be positive, but in other instances some of those weights will be negative. First, one can show that for any \((g,t)\in\{0,...,\bar{g}\}\times\{1,...,\bar{t}\}\),

\[
es_{2,g,t} = E(D_{g,t}) - E(D_{g,t-1}) - (E(D,.t) - E(D,.t-1)).\]

Moreover, under Assumptions 1-3, \(\beta_2\) well defined implies \(E(s_{G,T}e_{2,G,T}|S) > 0\). Therefore, the \((P(G = g, T = t|S)\omega_{2,g,t})_{g,t}\) are all positive if and only if for all \(t \geq 1\), in all the groups where the share of treated units strictly increases (resp. strictly decreases) between \(t - 1\) and \(t\), that increase (resp. decrease) is higher (resp. lower) than \(E(D,.t) - E(D,.t-1)\), the average evolution of the treatment rate across all groups between \(t - 1\) and \(t\). For instance, if for some \(t\) \(E(D_{g,t}) - E(D_{g,t-1})\) is strictly positive (resp. strictly negative) for all \(g\), but not constant across \(g\), then some of the \((P(G = g, T = t|S)\omega_{2,g,t})_{g,t}\) are negative. Second, when \(G \perp T\), meaning that the distribution of groups is stable over time, one can show that \(\varepsilon_{3,g,1} = (f(g) - E(f(G)))(1 - E(T)).\) Moreover, under Assumptions 1-3, \(\beta_3\) well defined implies \(E(s_{G,1}\varepsilon_{3,G,1}|S) > 0\). Therefore, the \((\omega_{3,g,1}P(G = g, T = 1S))_{g,t}\) are all positive if and only if all the groups where the share of treated units strictly increases (resp. strictly decreases) between \(T = 0\) and \(T = 1\) also have a value of \(f(g)\) higher (resp. lower) than \(E(f(G))\). Finally, it is more difficult to derive a simple necessary and sufficient condition under which the \((P(G = g, T = t|S)\omega_{1,g,t})_{g,t}\) are all positive. However, in Subsection 3.2, we derive such a condition in two special cases.

Theorem 1 shows that under Assumption 1 (resp. Assumptions 1-3) alone, \(\beta_1, \beta_2,\) and \(\beta_3\) do not identify \(\Delta^{TR}(S)\) (resp. \(\Delta^S(S)\)), and may not even satisfy the no-sign reversal property. We now provide conditions under which they do identify \(\Delta^{TR}(S)\) (resp. \(\Delta^S(S)\)).

**Assumption 4** \(_k\) (Random heterogeneity across the ATTs) \(\text{cov}(W_k, \Delta^{TR}_{G,T}|D = 1) = 0\).

**Assumption 5** \(_k\) (Random weights) \(\text{cov}(\Omega_k, \Delta^S_{G,T}|S) = 0\).

Assumptions 4\(_k\) and 5\(_k\) are indexed by \(k\), because the assumption one needs to invoke for identification depends on whether one considers \(\beta_1, \beta_2,\) or \(\beta_3\). Assumptions 4\(_1\), 4\(_2\), and 4\(_3\) (resp. Assumptions 5\(_1\), 5\(_2\), and 5\(_3\)) will automatically hold if \(\Delta^{TR}_{G,T}\) (resp. \(\Delta^S_{G,T}\)) is constant, meaning that the ATTs (resp. LATEs) are homogeneous across groups and time periods. If the ATTs (resp. LATEs) are heterogeneous across groups and time periods, Assumptions 4\(_k\) (resp. Assumptions 5\(_k\)) will still hold if \(W_k\) (resp. \(\Omega_k\)) is not systematically correlated to \(\Delta^{TR}_{G,T}\) (resp. \(\Delta^S_{G,T}\)).

**Corollary 1** Assume that \(D\) is binary, \(k \in \{1,2\}\) or \(k = 3\) and \(\bar{t} = 1\) and \(\beta_k\) is well defined.

\(^5\)For \((w_{2,g,t})_{g,t}\), this follows from \(E(\hat{w}_{2,G,T}) = 0\). For \((w_{3,g,t})_{g,t}\), this follows from a similar argument as for \((w_{1,g,t})_{g,t}\).
1. If Assumptions 1 and $4_k$ hold, $\beta_k = \Delta^{TR}$.

2. If Assumptions 1-3 and $5_k$ hold, $\beta_k = \Delta^S$.

This result follows directly from Theorem 1 and $E(W_k|D = 1) = E(\Omega_k|S) = 1$. Specifically, under Assumption 1 (resp. under Assumptions 1-3), we have $\beta_k - \Delta^{TR} = \text{cov}(W_k, \Delta^{TR}_G|D = 1)$ (resp. $\beta_k - \Delta^S = \text{cov}(\Omega_k, \Delta^S_{G,T}|D = 1)$). Therefore, if one is ready to further impose Assumption $4_k$ (resp. Assumption $5_k$), $\beta_k$ identifies $\Delta^{TR}$ (resp. $\Delta^S$).

We now discuss the plausibility of Assumptions 4, 4, 5, and 5. To simplify the discussion, let us momentarily assume that $T = 1$ and $G \perp T$. Then $\beta_1 = \beta_2$, so Assumptions 4 and 4 are equivalent, and Assumptions 5 and 5 are also equivalent. First, Assumptions 4 and 4 hold if $\text{cov}(\varepsilon_{1,G,T}, \Delta^{TR}_{G,T}|D = 1) = 0$. This is implausible. Positive (resp. negative) values of $\varepsilon_{1,g,t}$ correspond to values of $(g,t)$ for which the share of treated units is larger (resp. lower) than predicted by the regression of $E(D_{g,t})$ on group and period fixed effects. Those may also be the values of $(g,t)$ with the largest treatment effect. For instance, one can show that if selection into treatment is determined by a Roy model, then $\varepsilon_{1,g,t}$ and $\Delta^{TR}_{g,t}$ are positively related when $G \perp T$.\footnote{Assume that $D = 1\{Y(1) - Y(0) > 0\}, Y_{g,t}(1) - Y_{g,t}(0) \sim N(\mu_{g,t}, 1)$, and $G \perp T$. Then, one can show that $\frac{\partial \varepsilon_{1,g,t}}{\partial \mu_{g,t}} > 0$ and $\frac{\partial \Delta^{TR}_{g,t}}{\partial \mu_{g,t}} > 0$. Therefore, $\varepsilon_{1,g,t}$ and $\Delta^{TR}_{g,t}$ are positively related.}

Second, Assumptions 5 hold if $\text{cov}(\varepsilon_{1,G,T}, \Delta^{S}_{G,T}) = 0$. This will hold if $P(s_{G,1} = 1|S) = P(s_{G,1} = -1|S)$ and $s_{G,1} \perp (\varepsilon_{1,G,1}, \Delta^{S}_{G,1})|S$. However, $P(s_{G,1} = 1|S) = P(s_{G,1} = -1|S)$ implies $P(D = 1|T = 0) = P(D = 1|T = 1)$. This latter equality is testable, and it will often be rejected in the data. When it is not, one may still worry that $s_{G,T}$ is not independent of $\Delta^{S}_{G,T}$: groups where the share of treated units decreases may also be those where switchers experience a decrease of their treatment effect. This may result in a positive correlation between $s_{G,1}$ and $\Delta^{S}_{G,1}$.

We then discuss the plausibility of Assumptions 4 and 5. To simplify the discussion of Assumption 4, let us momentarily assume that $G \perp T, D \perp T$, and $\Delta^{TR}_{G,1} = \Delta^{TR}_{G,0} = \Delta^{TR}_G$. The two latter conditions require that the treatment rate in the population be stable between $T = 0$ and $T = 1$, and that the ATT in each group remains stable over time. Then, one can show that

$$\text{cov}(\varepsilon_{3,G,T}, \Delta^{TR}_{G,T}|D = 1) = P(T = 1)P(T = 0) \sum_{g=0}^{g} \frac{\partial (f(g) - E(f(G)))}{\partial \Delta^{TR}_{g}}(P(G = g|D = 1, T = 1) - P(G = g|D = 1, T = 0)).$$

As $f(G)$ typically is the intensity of an incentive for treatment administered at the group level in period 1, groups with the largest value of $f(G)$ should also be those where the share of treated units increases the most between $T = 0$ and $T = 1$. If those groups tend to have lower (resp.
higher) ATTs than the average, the rhs of the previous display will be strictly negative (resp. strictly positive). In both cases, Assumption 4\(_3\) will fail. Obviously, \(D \perp T\) is rarely met, while \(\Delta_{G,T}^{TR} = \Delta_{G,0}^{TR} = \Delta_{G}^{TR}\) may be implausible. But if Assumption 4\(_3\) is likely to fail when those two assumptions hold, it is still unlikely to hold when they fail. Second, Assumption 5\(_3\) holds if \(\text{cov} (s_{G,1} \varepsilon_{3,G,1}, \Delta_{G,1}^{S}) = 0\). This will hold if \(P(s_{G,1} = 1|S) = P(s_{G,1} = -1|S)\) and \(s_{G,1} \perp (\varepsilon_{3,G,1}, \Delta_{G,1}^{S})|S\). But as discussed above, \(P(s_{G,1} = 1|S) = P(s_{G,1} = -1|S)\) will often be rejected in the data, and when it is not one may still worry that \(s_{G,T}\) is not independent of \(\Delta_{G,T}^{S}\).

Finally, note that Assumptions 4\(_k\) and 5\(_k\) are partly testable. First, if \(\beta_1, \beta_2,\) and \(\beta_3\) are statistically different, under Assumption 1 (resp. Assumptions 1-3), one can reject Assumption 4\(_k\) (resp. Assumption 5\(_k\)) for at least two values of \(k \in \{1, 2, 3\}\). Second, assume that researchers observe a variable \(P_{g,t}\) that is likely to be correlated with the intensity of the treatment effect across groups and time periods. Then, they can run a suggestive test of Assumption 4\(_k\) (resp. Assumption 5\(_k\)), by testing whether \(\text{cov}(W_k, P_{G,T}|D = 1) = 0\) (resp. \(\text{cov}(\Omega_k, P_{G,T}|S) = 0\)).

In Section 6, we conduct this suggestive test in several applications, and find that it is often rejected.

When the treatment effect is heterogeneous across groups and periods, \(\beta_k\) may be a misleading measure of \(\Delta_{G,T}^{TR}\) (resp. \(\Delta_{G}^{S}\)) if Assumption 4\(_k\) (resp. 5\(_k\)) fails. In the corollary below, we derive the minimum amount of heterogeneity of the \((\Delta_{g,t}^{TR})_{g,t}\) (resp. \((\Delta_{g,t}^{S})_{g,t}\)) that could actually lead \(\beta_k\) to be of a different sign than \(\Delta_{G,T}^{TR}\) (resp. \(\Delta_{G}^{S}\)). Let \(\sigma_{TR}^{2} = V (\Delta_{G,T}^{TR}|D = 1)^{1/2}\) and \(\sigma_{S}^{2} = V (\Delta_{G,T}^{S}|S)^{1/2}\).

**Corollary 2** Assume that \(D\) is binary, \(k \in \{1, 2\}\) or \(k = 3\) and \(\bar{t} = 1\), and \(\beta_k\) is well defined.

1. Suppose that Assumption 1 holds and \(V(W^k|D = 1) > 0\). Then, the minimal value of \(\sigma_{TR}^{2}\) compatible with \(\beta_k\) and \(\Delta_{G,T}^{TR} = 0\) is

\[
\sigma_{TR}^{k} = \frac{|\beta_k|}{V(W^k|D = 1)^{1/2}}.
\]

2. Suppose that Assumptions 1-3 hold and \(V(\Omega^k|S) > 0\). Then, the minimal value of \(\sigma_{S}^{2}\) compatible with \(\beta_k\) and \(\Delta_{G}^{S} = 0\) is

\[
\sigma_{S}^{k} = \frac{|\beta_k|}{V(\Omega^k|S)^{1/2}}.
\]

Estimators of \(\sigma_{TR}^{k}\) (resp. \(\sigma_{S}^{k}\)) can be used to assess the sensitivity of \(\beta_k\) to treatment effect heterogeneity across groups and periods when Assumption 4\(_k\) (resp. 5\(_k\)) fails. Assume for instance that \(\beta_1\) is large and positive, while \(\sigma_{TR}^{1}\) is close to 0. Then, even under a minor amount of treatment effect heterogeneity, \(\beta_1\) could be of a different sign than \(\Delta_{G,T}^{TR}\). Indeed, the data
is compatible with $\beta_1$ large and positive, $\Delta^{TR} = 0$, and a small dispersion of the $(\Delta^{TR}_{g,t})_{g,t}$. To assess whether $\sigma_k^{TR}$ and $\sigma_k^S$ are small or large, we recommend to divide them by $\beta_k$. For instance, a standard deviation of the $(\Delta^{TR}_{g,t})_{g,t}$ equal to, say, 20% of $\beta_k$, can arguably be considered as small, as under Assumption 4, $\beta_k$ is equal to the average of $(\Delta^{TR}_{g,t})_{g,t}$ conditional on $D = 1$.

The first (resp. second) point of Corollary 2 does not apply when $V(W^k|D = 1) = 0$ (resp. $V(\Omega^k|S) = 0$). In such a case, $\beta_k = \Delta^{TR}$ (resp. $\beta_k = \Delta^S$) under Assumption 1 (resp. Assumptions 1-3) alone, even if the $(\Delta^{TR}_{g,t})_{g,t}$ (resp. $(\Delta^S_{g,t})_{g,t}$) are heterogeneous. However, the next proposition shows that in practice, it will rarely be the case that $V(W^k|D = 1) = 0$ (resp. $V(\Omega^k|S) = 0$).

**Proposition 1** Suppose that $D$ is binary, $k \in \{1, 2\}$ or $k = 3$ and $\bar{t} = 1$, and $\beta_k$ is well-defined. Then:

1. $V(W^k|D = 1) = 0$ implies that (i) for all $g$, there exists $t$ such that $E(D_{g,t}) = 0$ and (ii) for all $t$, there exists $g$ such that $E(D_{g,t}) = 0$.
2. $V(\Omega^k|S) = 0$ implies that for all $t \in \{1, \ldots, \bar{t}\}$, $g \mapsto s_{g,t}$ is not constant and equal to either -1 or 1.

The conditions above are necessary but not sufficient for the weights in Theorem 1 to be constant. The first condition will be violated whenever there is a group that is at least partly treated at every period, or a period at which all groups are at least partly treated. The second condition will be violated whenever there are two consecutive periods between which the mean treatment strictly increases in every group, or strictly decreases in every group.

### 3.2 The weights in two particular designs

We now consider two pervasive research designs in which one can derive easily interpretable necessary and sufficient conditions to have that all the weights in Theorem 1 are positive. First, we consider the staggered adoption design.

**Assumption 6** *(Staggered adoption)* $D = 1\{T \geq a_G\}$ for some $a_G \in \{0, \ldots, \bar{t}, \bar{t} + 1\}$.

**Assumption 7** *(Group × period regression with a balanced panel of groups)* For every $(g, t) \in \{0, \ldots, \bar{g}\} \times \{0, \ldots, \bar{t}\}$, $P(G = g, T = t) = \frac{1}{\bar{g}+1} \frac{1}{\bar{t}+1}$.

Assumption 6 is satisfied when each group is either fully untreated at each period, fully treated at each period, or fully untreated until a period $a_g - 1$ and fully treated from period $a_g$ onwards. Groups $g$ with $a_g = \bar{t} + 1$ never adopt the treatment. This type of staggered adoption design is often met in practice, see e.g. Athey and Stern (2002). In such instances, the data typically
has strictly more than two time periods, so researchers usually estimate \( \beta_1 \) or \( \beta_2 \). Hereafter, we refer to \( g(e) = \arg\min_{g \in \{0, \ldots, \gamma\}} a(g) \) as the earliest adopting group. Assumption 7 is for instance satisfied when regression 1, 2, or 3 is estimated with only one unit per group and period, and when no group appears or disappears over time. It is necessary to obtain some, but not all the results in Proposition 2 below.

**Proposition 2** Suppose that Assumption 6 holds, and that \( \beta_1 \) and \( \beta_2 \) are well defined. Then,

1. If Assumption 7 also holds, \( P(W_1 \geq 0 | D = 1) = 1 \) and \( P(\Omega_1 \geq 0 | S) = 1 \) if and only if \[ \frac{a_{g(e)}}{\bar{t} + 1} \geq E(D,\bar{t}) - E(D). \]

2. If Assumption 7 also holds, \( P(W_2 \geq 0 | D = 1) = 1 \) if and only if \( a_g \geq \bar{t} \) for every \( g \in \{0, \ldots, \gamma\} \).

3. \( P(\Omega_2 \geq 0 | S) = 1 \).

The first point of Proposition 2 shows that in the staggered-adoption research design, the weights \( (P(G = g, T = t|D = 1)w_{1,g,t})_{g,t} \) and \( (P(G = g, T = t|S)\omega_{1,g,t})_{g,t} \) are all positive if and only if the proportion of periods during which the earliest adopting group remains untreated is larger than the difference between the proportion of groups treated at \( \bar{t} \) and the average of that proportion across all periods. This condition is rarely met. For instance, it fails when at least one group is treated at \( T = 0 \). The second point of Proposition 2 then shows that in this research design, the weights \( (P(G = g, T = t|D = 1)w_{2,g,t})_{g,t} \) are all positive if and only if every group either remains untreated or becomes treated at the last period. This condition is also rarely met. Finally, the third point of Proposition 2 shows that in this research design, the weights \( (P(G = g, T = t|S)\omega_{2,g,t})_{g,t} \) are always all positive. Contrary to the first and second point, this result does not rely on Assumption 7.

Overall, in staggered adoption designs, Assumption 1 is not sufficient for \( \beta_1 \) and \( \beta_2 \) to satisfy the no-sign reversal property. On the other hand, if Assumption 7 holds, \( \beta_2 \) satisfies the no-sign reversal property if Assumptions 1 and 3 hold,\(^9\) while \( \beta_1 \) satisfies that property if Assumptions 1 and 4\(_1\) hold. Assumption 3 requires that the treatment effect be stable over time in each group, but it does not restrict the treatment effect heterogeneity across groups. On the other hand, Assumption 4\(_1\) restricts the treatment effect heterogeneity both over time and across groups. Therefore, in staggered adoption designs where it is plausible that the treatment effect is homogenous over time but not across groups, one may prefer to estimate \( \beta_2 \) rather than \( \beta_1 \).

The second design we consider is the heterogeneous adoption design.

---

8We then have \( a_{g(e)}/(\bar{t} + 1) = 0 \), while \( E(D,\bar{t}) - E(D) > 0 \), otherwise \( \beta_1 \) would not be well defined.

9Under Assumptions 7 and 6, Assumption 2 automatically holds.
Assumption 8 (Heterogeneous adoption) \( T = 1 \), \( D \) is binary, and for every \( g \in \{0, ..., \bar{g}\} \), \( E(D_{g,0}) = 0 \) and \( E(D_{g,1}) > 0 \).

Assumption 8 is satisfied in applications with two time periods, where all groups are fully untreated at \( T = 0 \), and where all groups become at least partly treated at \( T = 1 \). This type of heterogeneous adoption design is often met in practice, see, e.g., Enikolopov et al. (2011).

Proposition 3 Suppose that Assumption 8 holds, and that for every \( k \in \{1, 2, 3\} \), \( \beta_k \) is well defined. Then, \( P(W_k < 0|D = 1) > 0 \) and \( P(\Omega_k < 0|S) > 0 \).

Proposition 3 shows that in the heterogeneous adoption design, for every \( k \in \{1, 2, 3\} \) some of the weights \( P(G = g, T = t|D = 1)\omega_{k,g,t}\) and \( P(G = g, T = t|S)\omega_{k,g,t}\) are strictly negative. Then, the causal interpretation of \( \beta_k \) heavily relies on Assumption 4 or 5.

4 Extensions

4.1 Non-binary, ordered treatment

We now consider the case where the treatment takes a finite number of ordered values, \( D \in \{0, 1, ..., \bar{d}\} \). To accommodate for this extension, we have to modify Assumption 3 as follows.

Assumption 3O (Homogeneous ATTs across periods for ordered D) For all \( (d, g, t) \in \{1, ..., \bar{d}\} \times \{0, ..., \bar{g}\} \times \{1, ..., \bar{T}\} \), \( E(Y(d) - Y(0)|G = g, T = t, D(t - 1) = d) = E(Y(d) - Y(0)|G = g, T = t - 1, D(t - 1) = d) \).

We also need to modify the treatment effect parameters we consider. In lieu of \( \Delta^{TR} \), we consider \( \text{ACR}^{TR} = E(Y(D) - Y(0))/E(D) \), the average causal response (ACR) on the treated. For any \( (g, t) \in \{0, ..., \bar{g}\} \times \{0, ..., \bar{T}\} \), let \( \text{ACR}_{g,t}^{TR} = E(Y_{g,t}(D) - Y_{g,t}(0))/E(D_{g,t}) \) denote the ACR in group \( g \) and at period \( t \). Let us also define the probability measure \( P^{TR} \) by \( P^{TR}(A) = E(D_1A)/E(D) \) for any measurable set \( A \). This probability measure generalizes the conditional probability \( P(.|D = 1) \) for a binary treatment to non-binary treatments. We let \( E^{TR} \) and \( \text{cov}^{TR} \) denote the expectation and covariance operators associated to \( P^{TR} \). Then note that

\[
\text{ACR}^{TR} = E^{TR}[\text{ACR}^{TR}_{G,T}].
\]

In lieu of \( \Delta^S \), we consider \( \text{ACR}^S = E[\text{ACR}^S_{G,T}|S] \), where for all \( (g, t) \in \{0, ..., \bar{g}\} \times \{1, ..., \bar{T}\} \),

\[
\text{ACR}^S_{g,t} = \sum_{d=1}^{\bar{d}} \frac{P(D_{g,t} \geq d) - P(D_{g,t-1} \geq d)}{E(D_{g,t}) - E(D_{g,t-1})} \Delta^S_{dgt},
\]

\[
\Delta^S_{dgt} = E[Y_{g,t}(d) - Y_{g,t}(d - 1)|\max(D(t), D(t - 1)) \geq d > \min(D(t), D(t - 1))].
\]
\( \Delta_{dg}^S \) is the average effect of going from \( d - 1 \) to \( d \) units of treatment, in period \( t \), and among the units in group \( g \) whose treatment switches from strictly below to above \( d \) or from above to strictly below \( d \) between \( T = t - 1 \) and \( T = t \). \( \text{ACR}_{g,t}^S \) is a weighted average over \( d \) of the \( \Delta_{dg}^S \), where each of the \( \Delta_{dg}^S \) is weighted proportionally to the size of the corresponding population of switchers. \( \text{ACR}_{g,t}^S \) is similar to the ACR parameter considered in Angrist and Imbens (1995). \( \text{ACR}^S \) is a weighted average of the \( \text{ACR}_{g,t}^S \) across groups and periods.

Finally, we generalize Assumptions 4 and 5 as follows.

**Assumption 4O** (Random weights) \( \text{cov}^{TR}(W_k, \text{ACR}^{TR}_{G,T}) = 0. \)

**Assumption 5O** (Random weights) \( \text{cov}(\Omega_k, \text{ACR}^S_{G,T}|S) = 0. \)

Theorem 2 below generalizes Theorem 1 and Corollary 1 to the case where the treatment takes a finite number of ordered values.

**Theorem 2** Assume that \( D \in \{0, ..., \overline{d}\} \), \( k \in \{1, 2\} \) or \( k = 3 \) and \( t = 1 \), and \( \beta_k \) is well-defined.

1. If Assumption 1 holds, then
   \[
   \beta_k = E^{TR}[W_k \text{ACR}^{TR}_{G,T}].
   \]
   If Assumption 4O further holds, then \( \beta_k = \text{ACR}^{TR}. \)

2. If Assumptions 1-2 and 3O hold, then
   \[
   \beta_k = E[\Omega_k \text{ACR}^S_{G,T}|S].
   \]
   If Assumption 5O further holds, then \( \beta_k = \text{ACR}^S. \)

The first (resp. second) point of Theorem 2 shows that under Assumption 1 (resp. Assumption 1-2 and 3O), \( \beta_1, \beta_2, \) and \( \beta_3 \) identify weighted sums of the \( (\text{ACR}^{TR}_{g,t})_{g,t} \) (resp. \( (\text{ACR}^S_{g,t})_{g,t} \)), where the weights are the same as those in Theorem 1. Therefore, without further restrictions on \( (\text{ACR}^{TR}_{g,t})_{g,t} \) (resp. \( (\text{ACR}^S_{g,t})_{g,t} \)), \( \beta_1, \beta_2, \) and \( \beta_3 \) do not satisfy the no-sign reversal property in general. For every \( k \in \{1, 2, 3\} \), under Assumption 4O (resp. 5O), \( \beta_k \) identifies \( \text{ACR}^{TR} \) (resp. \( \text{ACR}^S \)). Yet, as already explained in the binary treatment case, these assumptions may not be plausible.

Theorem 2 extends to a continuous treatment. In such instances, one can for instance show that under Assumptions 1-2 and an appropriate generalization of Assumption 3, \( \beta_1, \beta_2, \) and \( \beta_3 \) identify weighted sums of the same weighted averages of the derivative of potential outcomes with respect to treatment in group \( g \) and at time \( t \) as in Angrist et al. (2000).

Finally, given that the weights in Theorem 2 are the same as in Theorem 1, and because the proof of Corollary 2 does not rely on the nature of the treatment, Corollary 2 directly applies to ordered treatments.
4.2 Including covariates

Often times, researchers also include a vector of covariates $X$ as control variables in their regression. We show in this section that our results can be extended to this case. We start by redefining the three regressions we consider in this context.

Regression 1X

Let $\beta_1^X$ and $\gamma_1$ denote the coefficients of $E(D|G,T)$ and $X$ in an OLS regression of $Y$ on a constant, $(1\{G = g\})_{1 \leq g \leq G}$, $(1\{T = t\})_{1 \leq t \leq T}$, $E(D|G,T)$, and $X$.

Regression 2X

Let $\beta_2^X$ and $\gamma_2$ denote the coefficients of $FD_D(G,T)$ and $FD_X(G,T)$ in an OLS regression of $FD_Y(G,T)$ on a constant, $(1\{T = t\})_{2 \leq t \leq T}$, $FD_D(G,T)$, and $FD_X(G,T)$, conditional on $T \geq 1$.

Regression 3X

If $t = 1$, let $\beta_3^X$ denote the coefficient of $D$ in a 2SLS regression of $Y$ on a constant, $(1\{G = g\})_{1 \leq g \leq G}$, $1\{T = 1\}$, $D$, and $X$, where $D$ is instrumented by $f(G)1\{T = 1\}$. Let also $\gamma_3$ (resp. $\gamma_4$) denote the coefficient of $X$ in a regression of $Y$ (resp. $D$) on a constant, $(1\{G = g\})_{1 \leq g \leq G}$, $1\{T = 1\}$, $f(G)1\{T = 1\}$, and $X$.

Then, we need to modify Assumption 1 as follows.

Assumption 1X$_k$ (Common trends for $\beta^X_k$) $E(Y(0) - X'\gamma_k|G,T = t) - E(Y(0) - X'\gamma_k|G,T = t - 1)$ does not depend on $G$ for all $t \in \{1,...,T\}$. If $k = 3$, we also have $cov(X,\varepsilon_{3,G,T}) = 0$.

Assumptions 1X$_1$, 1X$_2$, and 1X$_3$ are implied by the linear and constant treatment effect models that are often invoked to justify the use of Regression 1X, 2X, and 3X. For instance, the use of Regression 1X is often justified by the following model:

$$Y = \gamma_G + \lambda_T + \theta D + X'\gamma_1 + \varepsilon, \quad E(\varepsilon|G,T,D,X) = 0. \quad (6)$$

Equation (6) implies Assumption 1X$_1$, but it does not imply Assumption 1. This is why we consider the former common trends assumption instead of the latter in this subsection. Similarly, the linear and constant treatment effect model rationalizing the use of Regression 2X and 3X respectively imply Assumption 1X$_2$ and the first condition in Assumption 1X$_3$.

Assumption 1X$_1$ requires that once netted out from the partial linear correlation between the outcome and $X$, $Y(0)$ satisfies the common trends assumption. It may be more plausible than Assumption 1, for instance if there are group-specific trends affecting the outcome but if those group-specific trends can be captured by a linear model in $X$. Similar interpretations apply to Assumption 1X$_2$ and the first condition in Assumption 1X$_3$. 

17
Finally, we need to assume that the treatment has no effect on the covariates. Let \( X(0) \) and \( X(1) \) respectively denote the potential covariates of the same unit without and with the treatment.

**Assumption 9 (No treatment effect on the covariates)** \( X(0) = X(1) = X. \)

Assumption 9 will for instance hold if \( X \) is determined prior to the treatment.

Theorem 3 below generalizes Theorem 1 to the case where there are covariates in the regression.

**Theorem 3** Assume that \( D \) is binary, \( k \in \{1, 2\} \) or \( k = 3 \) and \( \bar{t} = 1 \), and \( \beta_k \) is well-defined.

1. If Assumptions 1\( X_k \) and 9 hold, then

\[
\beta^X_k = E \left[ W_k \Delta^T_{G,T} | D = 1 \right].
\]

If Assumption 4\( k \) further holds, then \( \beta^X_k = \Delta^T. \)

2. If Assumptions 1\( X_k \), 2, 3, and 9 hold, then

\[
\beta^X_k = E \left[ \Omega_k \Delta^S_{G,T} | S \right].
\]

If Assumption 5\( k \) further holds, then \( \beta^X_k = \Delta^S. \)

Theorem 3 shows that under a modified version of the common trends assumption accounting for the covariates, for every \( k \in \{1, 2, 3\} \) \( \beta^X_k \) identifies the same weighted sum of the \( (\Delta^T_{g,t})_{g,t} \) as in Theorem 1. If one further imposes Assumptions 2 and 3, \( \beta^X_k \) identifies the same weighted sum of the \( (\Delta^S_{g,t})_{g,t} \) as in Theorem 1. Therefore, if the regression without covariates has some negative weights and one worries that treatment effect heterogeneity may be systematically related to the weights, the addition of covariates will not alleviate that concern.

Note that when \( k = 3 \), Theorem 3 is derived under a common trends assumption and the condition that \( \text{cov}(X, \varepsilon_{3,G,T}) = 0 \). This condition can be assessed from the data. If it is not satisfied, the weights have to be modified, and their average is not equal to 1 anymore. For instance, one can see from the proof of Theorem 3 that if the first statement of Assumption 1\( X_3 \) and Assumption 9 hold but \( \text{cov}(X, \varepsilon_{3,G,T}) \neq 0 \), then

\[
\beta_k = \sum_{(g,t)} \frac{P(G = g, T = t) E(D_{g,t} | \varepsilon_{3,g,t})}{\sum_{g',t'} P(G = g, T = t) E(D_{g,t} - X_{g,t}^4 | \varepsilon_{3,g',t'})} \Delta^T_{g,t}.
\]

### 5 Alternative estimands

In this section, we show that \( \Delta^S \) is actually identified without relying on the random heterogeneity conditions, under the following testable condition on the design.
**Assumption 10 (Existence of “stable” groups)** For all \(t \in \{1, \ldots, \bar{t}\}\), there exists \(g \in \{0, \ldots, \bar{g}\}\) such that \(E(D_{g,t}) = E(D_{g,t-1})\).

Assumption 10 is often satisfied. For instance, in the staggered adoption design, for all \(t\) there must be a group whose exposure to the treatment does not change between \(t-1\) and \(t\). Indeed, the converse condition would imply that all the groups adopt the treatment at the same period, but then \(\beta_1\) and \(\beta_2\) would not be well defined. It is also satisfied in Gentzkow et al. (2011): between each pair of successive elections, there are counties where the number of newspapers available does not change.

For every \(t \in \{1, \ldots, \bar{t}\}\), let us introduce the “supergroup” variable \(G_t^* = s_{G,t}\). For instance, groups with \(G_t^* = 1\) are those where the mean treatment strictly increases between \(T = t - 1\) and \(T = t\). For all \((d, g, g*, t, t') \in \{0, 1\} \times \{0, \ldots, \bar{g}\} \times \{-1, 0, 1\} \times \{1, \ldots, \bar{t}\}\), let \(r(g|g*, t', t) = P(G = g|G_t^* = g*, T = t), r_d(g'|t, t) = P(G = g|G_t^* = 0, T = t, D = d)\), and let

\[
Q = \frac{r(G|G_{t+1}^*, T + 1, T + 1)}{r(G|G_{t+1}^*, T + 1, T)}, \quad Q_d = \frac{r_d(G|T + 1, T + 1)}{r_d(G|T + 1, T)}.
\]

The ratio \(r(g|g*, t + 1, t + 1)/r(g|g*, t + 1, t)\) is the share of group \(g\) in the supergroup \(G_t^* = g^*\) at \(T = t + 1\), divided by the share of group \(g\) in the supergroup \(G_t^* = g^*\) at \(T = t\). Therefore, \(Q\) is larger (resp. lower) than 1 for units whose group’s size has increased faster (resp. more slowly) than their period-\(T + 1\) super-group’s size between \(T = t\) and \(T = t + 1\). \(Q_d\) can be interpreted similarly, but conditional on \(D = d\). Note that if \(G \perp \perp T\), \(Q = Q_0 = Q_1 = 1\).

Before introducing our two estimands, let us define, for any random variable \(R\) and for all \((d, g, t) \in \{0, 1\} \times \{-1, 0, 1\} \times \{1, \ldots, \bar{t}\}\), the following quantities:

\[
DID_R^*(g, t) = E(R|G_t^* = g, T = t) - E(QR|G_t^* = g, T = t - 1) - (E(R|G_t^* = 0, T = t) - E(QR|G_t^* = 0, T = t - 1)),
\]

\[
\delta_d = E(Y|D = d, G_t^* = 0, T = t) - E(Q_dY|D = d, G_t^* = 0, T = t - 1)
\]

\[
TCD^*(g, t) = E(Y|G_t^* = g, T = t) - E(Q(Y + \delta_d)|G_t^* = g, T = t - 1).
\]

\(DID_R^*(g, t)\) compares the evolution of the mean of \(R\) in the supergroups \(G_t^* = g\) and \(G_t^* = 0\) between \(T = t - 1\) and \(T = t\), after reweighting units in period \(t - 1\) by \(Q\) to ensure that groups’ distribution is the same in periods \(t - 1\) and \(t\). \(TCD^*(g, t)\) stands for “time-corrected difference”. It compares the mean outcome in the supergroup \(G_t^* = g\) at \(T = t\) to the mean outcome in the same group at \(T = t - 1\), after imputing the trends on \(Y(0)\) and \(Y(1)\) observed in the supergroup \(G_t^* = 0\) to the period \(T = t - 1\) mean outcome in the supergroup \(G_t^* = g\). Here as well, the reweighting of units by \(Q_d\) and \(Q\) in period \(t - 1\) ensures that groups’ distribution is the same in periods \(t - 1\) and \(t\). Let also

\[
w_{g,t} = \frac{gDID_D^*(g, t)P(G_t^* = g, T = t)}{\sum_{t'=1}^{\bar{t}} \sum_{g' \in \{-1, 1\}} g'DID_D^*(g', t')P(G_t^* = g', T = t')}.
\]
Note that by construction, \( w_{g,t} \geq 0 \). Our estimands are defined by

\[
W_{DID} = \sum_{t=1}^{i} \sum_{g \in \{-1,1\}} w_{g,t} \text{DID}^*_D(g,t) / \text{DID}^*_D(g,t)
\]

\[
W_{TC} = \sum_{t=1}^{i} \sum_{g \in \{-1,1\}} w_{g,t} \text{TCD}^*_D(g,t) / \text{DID}^*_D(g,t)
\]

with the convention that \( w_{g,t} / \text{DID}^*_D(g,t) = 0 \) when \( P(G^*_t = g, T = t) = 0 \).

Theorem 4 below shows that \( W_{DID} \) and \( W_{TC} \) identify \( \Delta^S \) without relying on any random heterogeneity condition. The first estimand \( W_{DID} \) still relies on Assumption 3, which restricts treatment effect heterogeneity. The second estimand \( W_{TC} \) does not impose rely on any restriction on treatment effect heterogeneity. Instead, it relies on the following conditional common trends condition.

**Assumption 1’** (Common trends conditional on \( D(t-1) \)) For all \((d,t) \in \{0,1\} \times \{1,...,t\}\),

\[
E(Y(d) = d|D(t-1) = d, T = t, G) - E(Y(d)|D(t-1) = d, T = t-1, G) \text{ does not depend on } G.
\]

Assumption 1 imposes two conditional common trends conditions. First, it requires that for all \( t \geq 1 \), the evolution of the mean \( Y(0) \) of units untreated at \( T = t-1 \) be the same in every group. Then, it requires that for all \( t \geq 1 \), the evolution of the mean \( Y(1) \) of units treated at \( T = t-1 \) be the same in every group.

**Theorem 4** Suppose that \( D \) is binary and Assumption 10 holds.

1. If Assumptions 1-3 hold, then \( W_{DID} = \Delta^S \).
2. If Assumptions 1’ and 2 hold, then \( W_{TC} = \Delta^S \).

Theorem 4 shows that when “stable” groups exist for each pair of consecutive time periods, a weighted average of the Wald-DID (resp. Wald-TC) estimands proposed in de Chaisemartin and D’Haultfouillle (2017a) identifies \( \Delta^S \) under Assumptions 1-3 (resp. Assumptions 1’ and 2). The averaging of those estimands is close to that proposed in section 1.2 of de Chaisemartin and D’Haultfouillle (2017b), except that by reweighting observations at period \( t-1 \) and by considering slightly different weights \( w_{g,t} \), the current estimands identify \( \Delta^S \), instead of a less interpretable weighted average of switchers’ LATEs across periods.

Let us now give some intuition on the estimands \( W_{DID} \) and \( W_{TC} \). Under Assumptions 1-3, the evolution of the mean outcome in the “control group” \( (G^*_t = 0) \) identifies the evolution of the mean outcome we would have observed in the “treatment group” \( G^*_t = 1 \), if that group had not experienced any evolution of its treatment rate. Then, \( \text{DID}^*_D(1,t) \) identifies the treatment effect of the switchers in group \( G^*_t = 1 \) at \( T = t \), times the proportion of switchers. \( \text{DID}^*_D(1,t) \)
identifies the share of switchers in that group, so $DID^*(1, t)/DID_D^*(1, t)$ identifies the LATE of switchers in group $G^*_t = 1$ at $T = t$. Similarly, $DID^*(-1, t)/DID_D^*(-1, t)$ identifies the LATE of switchers in group $G^*_t = -1$ at $T = t$. Finally, $W_{DID}$ averages those estimands with weights equal to the proportion that each subgroup accounts for in the total population of switchers, so it identifies the LATE of all switchers.

Tuning to $W_{TC}$, it is worth noting that it has a simple expression when the treatment is constant within each group $\times$ period cell and $G \perp T$, as is for instance the case in Gentzkow et al. (2011). Then, let $NT_t = 1\{D_{G,t} = D_{G,t-1} = 0\}$ (resp. $AT_t = 1\{D_{G,t} = D_{G,t-1} = 1\}$) be an indicator for groups that are untreated (resp. treated) in periods $t - 1$ and $t$, the “never treated” (resp. “always treated”) groups. One can then show that

$$TCD^*(1, t)/DID_D^*(1, t) = E(Y|G^*_t = 1, T = t) - E(Y|G^*_t = 1, T = t - 1) - (E(Y|NT_t = 1, T = t) - E(Y|NT_t = 1, T = t - 1)), \nonumber$$

$$TCD^*(-1, t)/DID_D^*(-1, t) = E(Y|AT_t = 1, T = t) - E(Y|AT_t = 1, T = t - 1) - (E(Y|G^*_t = -1, T = t) - E(Y|G^*_t = -1, T = t - 1)). \nonumber$$

When the treatment is constant within each group $\times$ period cell, $TCD^*(1, t)/DID_D^*(1, t)$ is a DID estimand comparing the evolution of the mean outcome between groups that go from being untreated to treated between $t - 1$ and $t$, and groups that remain untreated between these two dates. Similarly, $TCD^*(-1, t)/DID_D^*(-1, t)$ is a DID estimand comparing the evolution of the mean outcome between groups that remain treated between $t - 1$ and $t$, and groups that go from being treated to untreated between these two dates.

When the treatment is not constant within each group $\times$ period cell or $G$ is not independent from $T$, the formula of $W_{TC}$ is more complicated, but here is the intuition underlying it. Under Assumption 1’, the trend affecting the $Y(0)$ (resp. $Y(1)$) of units with $D(t - 1) = 0$ (resp. $D(t - 1) = 1$) between $t - 1$ and $t$ is the same in every group. This trend is identified by the evolution of the mean of $Y$ of untreated (resp. treated) units between $t - 1$ and $t$ in all “stable” groups with $G^*_t = 0$: under Assumption 2, one must have $D(t - 1) = D(t)$ in those groups. Then, one can add the trend on $Y(0)$ (resp. $Y(1)$) to the outcome of untreated (resp. treated) units in group $G^*_t = 1$ in period $t - 1$, and thus recover the mean outcome we would have observed in this group in period $t$ if switchers had not changed treatment between the two periods. This is what $Y + \delta \rho$ does. Therefore, $TCD^*(1, t)$ compares the mean outcome in group $G^*_t = 1$ at $T = t$ to the counterfactual mean we would have observed in that group at $T = t$ if switchers had remained untreated. Following the same logic as for $W_{DID}$, $TCD^*(1, t)/DID_D^*(1, t)$ identifies the LATE of switchers in group $G^*_t = 1$ at $T = t$ and $W_{TC}$ identifies the LATE of all switchers.

It is worth noting that Theorem 4 can easily be extended to the case where the treatment is not binary, and to the case with covariates. Estimators of $W_{DID}$ and $W_{TC}$ are computed by the Stata package fuzzydid, see Chaisemartin et al. (2018).
6 Applicability, and applications

6.1 Applicability

We assess the pervasiveness of two-way fixed effects estimators in economics by conducting a review of all papers published in the American Economic Review (AER) between 2010 and 2012. Over these three years, the AER published 337 papers. This excludes papers and proceedings, comments, replies, and presidential addresses. Out of these 337 papers, 34 or 10.1% of them estimate Regressions 1, 2, or 3, or other two-way fixed effects regressions resembling closely Regressions 1, 2, or 3. When one withdraws from the denominator theory papers and lab experiments, the proportion of papers using these regressions raises to 19.5%. In the appendix, we review each paper and explain where it uses a two-way fixed effects estimator.

Table 1: Two-way fixed effects papers published in the AER (2010-2012)

<table>
<thead>
<tr>
<th></th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Papers using two-way fixed effects estimators</td>
<td>5</td>
<td>15</td>
<td>14</td>
<td>34</td>
</tr>
<tr>
<td>% of published papers</td>
<td>5.2%</td>
<td>13.0%</td>
<td>11.2%</td>
<td>10.1%</td>
</tr>
<tr>
<td>% of empirical papers, excluding lab experiments</td>
<td>12.8%</td>
<td>24.6%</td>
<td>19.2%</td>
<td>19.7%</td>
</tr>
</tbody>
</table>

6.2 Applications

Enikolopov et al. (2011)

Enikolopov et al. (2011) study the effect of NTV, an independent TV channel introduced in 1996 in Russia, on the share of the electorate voting for opposition parties. NTV’s coverage rate was heterogeneous across subregions: while a large fraction of the population received NTV in urbanized subregions, a smaller fraction received it in more rural subregions. The authors estimate Regression 1: they regress the share of votes for opposition parties in the 1995 and 1999 elections in Russian subregions on subregion fixed effects, an indicator for the 1999 election, and on the share of the population having access to NTV in each subregion at the time of the election. In 1995, the share of the population having access to NTV was equal to 0 in all subregions, while in 1999 it was strictly greater than 0 everywhere. Therefore, the authors’ research design corresponds exactly to the heterogenous adoption design considered in Subsection 3.2. Enikolopov et al. (2011) find that $\hat{\beta}_1 = 6.65$, with a standard error equal to
1.40. According to this regression, increasing the share of the population having access to NTV from 0 to 100% increases the share of votes for the opposition parties by 6.65 percentage points. Because $\bar{t} = 1$ and $T \perp G$, $\beta_1 = \beta_2$.

More than 50% of the weights attached to $\beta_1$ are negative. As no one was treated in 1995, the populations of treated units and switchers are the same, so the weights attached to $\beta_1$ under Assumption 1 and under Assumptions 1-3 are also the same. We estimate the weights $(P(G = g, T = 1|D = 1)w_{1,g,1})_g$. 918 are strictly positive, while 1,020 are strictly negative. The negative weights sum to -2.26 (t-stat =-38.96). Finally, we find that $\hat{\sigma}_{1TR}/\hat{\beta}_1 = 0.148$ (95% level confidence interval=[0.142, 0.153]). Namely, $\beta_1$ and $\Delta^{TR}$ may be of opposite signs if the standard deviation of the $(\Delta^{TR}_{g,t})_{g,t}$ is equal to 15% of $\beta_1$.

Therefore, the causal interpretation of $\beta_1$ relies on Assumption 4. This assumption is not warranted. For instance, we find that the weights $(P(G = g, T = 1|D = 1)w_{1,g,1})_g$ are strongly positively correlated with subregions’ population (t-stat=14.15). Accordingly, the population of regions receiving a positive weight is 46% larger than that of regions receiving a negative weight. The effect of NTV may be larger in less populated regions, as there, fewer other sources of information are available. Then, $\beta_1$ would underestimate the true effect of NTV.

**Gentzkow et al. (2011)**

Gentzkow et al. (2011) study the effect of newspapers on voters’ turnout in US presidential elections between 1868 and 1928. They regress the first difference of the turnout rate in county $g$ between election years $t$ and $t - 1$ on state-year fixed effects and on the first difference of the number of newspapers available in that county. This regression corresponds to Regression 2, with state-year fixed effects as controls. Gentzkow et al. (2011) find that $\hat{\beta}_2 = 0.0026$, with a standard error equal to 0.0009. According to this regression, one more newspaper increased voters’ turnout by 0.26 percentage points. Because $\bar{t} > 1$, $\hat{\beta}_1 \neq \hat{\beta}_2$. We find that $\hat{\beta}_1 = -0.0011$, with a standard error equal to 0.0011. $\hat{\beta}_1$ and $\hat{\beta}_2$ are significantly different (t-stat=3.64).

Around 50% of the weights attached to $\beta_1$ and $\beta_2$ under Assumption 1 are negative. We estimate the weights $(P(G = g, T = t|D = 1)w_{2,g,t})$: 4,002 are strictly positive, 6,376 are strictly negative. The negative weights sum to -1.28 (t-stat =-9.43). Similarly, around 40% of the weights $(P(G = g, T = t|D = 1)w_{1,g,1})_{g,t}$ are strictly negative. Therefore, under Assumption 1 the causal interpretation of $\beta_1$ and $\beta_2$ respectively relies on Assumption 4 and 4. Those two assumptions are not warranted: the fact that $\hat{\beta}_1$ and $\hat{\beta}_2$ significantly differ implies that at least one of them must be violated.

---

10 To draw inference on the estimators we compute in this subsection, we use the bootstrap, clustered at the subregion level.

11 To draw inference on the estimators we compute in this subsection, we use the bootstrap, clustered at the county level.
On the other hand we find that all the weights attached to $\beta_2$ under Assumptions 1-3 are positive, while 25% of the weights attached to $\beta_1$ are still negative. Therefore, under Assumptions 1-3, $\beta_2$ identifies a convex combination of causal effects even if switchers’ LATEs are heterogenous over time and across counties. On the other hand, even under those assumptions $\beta_1$ still relies on Assumption 5, which is not warranted.

Still, $\beta_2$ relies on Assumption 3, which may not be plausible in this context. Assumption 3 requires that in counties with at least one newspaper in election year $t - 1$, the effect of newspapers does not change between election years $t - 1$ and $t$. However, newspapers' readership systematically decreases between consecutive elections. On average across pairs of consecutive elections and counties, and restricting the sample to counties with at least one newspaper in the first of the two consecutive elections, the fraction of a county’s population reading the newspapers divided by the county’s number of newspapers decreases by 1.0 percentage point between two consecutive elections (t-stat=-6.70). As newspapers tend to be less widely read in election-year $t$ than in election-year $t - 1$, their effect may decrease between consecutive elections.

Assumption 10 holds in this application: between each pair of consecutive elections, there are counties where the number of newspapers did not change. We can then estimate $W_{TC}$, the estimand proposed in Section 5. Contrary to $\beta_2$, it does not rely on any restriction on treatment effect heterogeneity. Instead, it relies on Assumption 1’, a conditional common trends assumption.\(^\text{12}\) We find that $\hat{W}_{TC} = 0.0045$, with a standard error of 0.0016. $\hat{W}_{TC}$ is 73% larger than, and significantly different from, $\hat{\beta}_2$ (t-stat=1.98).

7 Conclusion

Almost 20% of empirical articles published in the AER between 2010 and 2012 use regressions with groups and period fixed effects to estimate treatment effects. While it is well-known that such regressions identify the treatment effect of interest if that effect is constant and if the standard common trends assumption is satisfied, those regressions have not yet been studied in a model allowing for treatment effect heterogeneity. In this paper, we start by showing that under the common trends assumption alone, three pervasive two-way fixed effects regressions identify weighted sums of the ATTs in each group and at each period. The weights can be estimated, and in most applications many weights are negative. Then, under the common trends assumption alone, two-way fixed effects regressions are not robust to heterogeneous treatment effects across groups and periods: the coefficient of the treatment variable in those regressions may for instance

\(^{12}\)de Chaisemartin and D'Haultfoeuille (2017b) conduct a placebo test of Assumption 1’. They find that conditional on their number of newspapers in $t-1$, counties with different evolutions of their number of newspapers from $t - 1$ to $t$ do not experience different evolutions of their turnout from $t - 2$ to $t - 1$. This suggests that Assumption 1’ is plausible in this application.
be negative while the treatment effect is positive for every unit in the population.

Then, we consider two supplementary assumptions. The first one requires that in each group and for each pair of consecutive periods, the average treatment effect of units treated at period \( t - 1 \) be stable from \( t - 1 \) to \( t \). The second one requires that in each group and for each pair of consecutive periods, the treatment follows a monotonic evolution from \( t - 1 \) to \( t \). Under the common trends assumption and those two supplementary assumptions, we show that our three two-way fixed effects regressions identify weighted sums of the LATEs of switchers in each group and at each period, where switchers are units whose treatment changes between two consecutive time periods. Here again, some of the weights may be negative. However, in some instances the weights attached to the second regression we consider, the first-difference regression, will all be positive. We also propose an estimator of the LATE of all switchers that does not rely on any treatment effect homogeneity condition and that can be used in applications where for each pair of consecutive time periods, there are groups whose exposure to treatment does not change between these dates.

Finally, we revisit two articles that have estimated two-way fixed effects regressions, and show that in both cases, more than half of the weights attached to their coefficient of interest under the common trend assumption are negative. In one of these two articles, we also find that our new estimator is economically and significantly larger than the estimator computed by the authors.
References


A Detailed literature review

We now review the 34 papers using two-way fixed effects estimators we found in our literature review. For each paper, we use the following presentation:

**Authors (year), Title.** Where the two-way fixed effects estimator is used in the paper.
Description of the two-way fixed effects estimator and how it relates to Regressions 1, 2 or 3, or to the Wald-DID estimator.

   The elasticity discussed after, say, Table 2 is estimated as the ratio of the effect of the Medicare reform on utilization, divided by the effect of the Medicare reform on co-payment. Both effects are estimated through standard DID regressions in Table 2. Therefore, the elasticity estimate is a Wald-DID.

   In regression equation (1), the dependent variable is the change in the price of drug j between 2003 and 2006, and the explanatory variable is the Medicare market share for drug j in 2003. This regression corresponds to Regression 2.

3. Aizer (2010), The Gender Wage Gap and Domestic Violence. *Table 2.*
   In regression equation (2), the dependent variable is the log of female assaults among females of race r in county c and year t, and the explanatory variables are race, year, county, race × year, race × county, and county × year fixed effects, as well as the gender wage gap in county c, year t, and race r. This regression is a “triple difference” version of Regression 1.

4. Algan and Cahuc (2010), Inherited Trust and Growth. *Figure 4 and Table 6.*
   Figure 4 presents a regression of changes in income per capita from 1935 to 2000 on changes in inherited trust over the same period and a constant. This regression corresponds to Regression 2.

5. Ellul et al. (2010), Inheritance Law and Investment in Family Firms. *Table 7.*
   In the regressions presented in Table 7, the dependent variable is the capital expenditure of firm j in year t, and the explanatory variables are firm fixed effects, an indicator for whether year t is a succession period for firm j, and the interaction of this indicator with the level of investor protection in the country where firm j is located. This regression corresponds to Regression 1, with two periods (succession and no succession).
In regression equation (11), the dependent variable is the change in exporting status of firm \( i \) in sector \( j \) between 1992 and 1996, and the explanatory variable is the change in trade tariffs in Brasil for products in sector \( j \) over the same period. This regression corresponds to Regression 2.

In the regression in Table 5 column (2), the dependent variable is an indicator for whether a car sold is a flexible fuel vehicle, and the explanatory variables are state and month fixed effects, and the percent ethanol availability in each month \( \times \) state. This regression corresponds to Regression 1.

In regression equations (15a) and (15b), the dependent variable is the ad valorem tariff level bound by country \( c \) on product \( g \), while the explanatory variables are country and product fixed effects, and two treatment variables which vary at the country \( \times \) product level. These regressions are similar to Regression 1, except that they have two treatment variables.

In the regression in, say, Table 3 column (4), the dependent variable is the total number of contributions to Wikipedia by individual \( i \) at period \( t \), regressed on individual fixed effects, an indicator for whether period \( t \) is after the Wikipedia block, and the interaction of this indicator and a measure of social participation by individual \( i \). This regression corresponds to Regression 1 (treatment is equal to 0 before the block, and to social participation after it).

In regression equation (7), the dependent variable is the change in crime rates between week \( t \) and the same week one year ago in borough \( b \), and the explanatory variables are an indicator for whether week \( t \) is around the terrorist attacks in London, and the number of police forces in borough \( b \) in week \( t \). The interaction of the time indicator and of whether borough \( b \) belongs to Theseus operation is used as the excluded instrument for police forces. This regression is equivalent to Regression 3 (borough fixed effects disappear...
because of the first differencing with respect to the previous year, something the authors do to control for seasonality).

11. Hotz and Xiao (2011), The Impact of Regulations on the Supply and Quality of Care in Child Care Markets. Table 7, Columns 4 and 5.
In Regression Equation (1), the dependent variable is the outcome for market m in state s and year t, and the explanatory variables are state and year fixed effects and a measure of regulations in state s in year t. This regression corresponds to Regression 1.

Regression equations (1) and (2) are first-difference versions of Regression 3. In levels, the instrument would be the elasticity interacted with the year 2006. Because the data only bears two periods, the two regressions are algebraically equivalent.

In regression equation (15), the dependent variable is a measure of the quantity of housing services in household i's residence in year t, while the explanatory variables are an indicator for period t being after the reform, a measure of mismatch in household i, and the interaction of the measure of mismatch and the time indicator. This regression is similar to Regression 1, except that it has a measure a mismatch in household i instead of household fixed effects.

In regression equation (4), the dependent variable is the change in vehicle kilometers traveled in MSA s between periods t and t-1, and the explanatory variable is the change in kilometers of roads in MSA s between periods t and t-1. This regression corresponds to Regression 2.

In regression equation (1), the dependent variable is urbanization in polity j at time t, while the explanatory variables are time and polity fixed effects, and the number of years of French presence in polity j interacted with the time effects. This regression corresponds to Regression 1.

In the regression presented in, say, the first column of Table 6, the dependent variable is enrolment in schools of MSA j in year t, while the explanatory variables are time and MSA effects and the value of the dissimilarity index of schools in MSA j in year t. The excluded instrument for the dissimilarity index is an indicator for whether in period t, the MSA was desegregated. This regression is similar to Regression 3.

17. Dinkelman (2011), The Effects of Rural Electrification on Employment: New Evidence from South Africa. Tables 4 and 5 columns 5-8, Table 8 columns 3-4, Table 9 column 2, and Table 10 columns 2, 4, and 6.
Regression equation (4) is the first-difference version of Regression 3. In levels, the instrument would be the land gradient $Z_j$ interacted with an indicator for the second wave of the panel. Because the data only bears two periods, the two regressions are algebraically equivalent.

18. Enikolopov et al. (2011), Media and Political Persuasion: Evidence from Russia. Table 3.
In regression equation (5), the dependent variable is the share of votes for party j in year t and subregion s, and the explanatory variables are subregion and time effects, and the share of people having access to NTV in subregion s in period t. This regression corresponds to Regression 1.

In regression equation (7), the dependent variable is the health expenditures of individual j working in industry i in period t and region r, and the explanatory variables are individual effects, region specific time effects, and the job tenure of individual j. The death rate of establishments in industry i in period t and region r is used as an instrument for the job tenure of individual j. Within each region, the regression has time effects and individual effects, and an instrument varying only across industry × periods. This regression is similar to Regression 3.

In regression equation (1), the dependent variable is the change in voter turnout in county c between elections year t and t-1, and the explanatory variables are state × year effects, and the number of newspapers in county c in year t. This regression corresponds to Regression 2, except that it allows for state specific trends.

In the regression in, say, column 6 of Table 2, the dependent variable is the log of output per worker in firm i in period t, while the explanatory variables are firms and time fixed effects, and the log of the amount of IT capital per employee (ln(C/L)) as well as the interaction of ln(C/L) and an indicator for whether the firm is owned by a US multinational. This regression is similar to Regression 1, except that it has two treatment variables.

In regression equation (5), the dependent variable is a measure of time to consensus for project i submitted to committee j, while the explanatory variables are an indicator for projects submitted to the standards track, a measure of distributional conflict, and the interaction of the standards track and distributional conflict. This regression is similar to Regression 1, except that it has a measure of distributional conflict instead of committee fixed effects.

In the regression equation in the beginning of Section III, the dependent variable is the number of patents by US inventors in patent class c at period t, and the explanatory variables are patent class and time fixed effects, and the interaction of period t being after the trading with the enemy act and a measure of treatment intensity. This regression corresponds to Regression 1 (treatment is equal to 0 before the act).

24. Forman et al. (2012), The Internet and Local Wages: A Puzzle. Tables 2 and 4.
In regression equation (1), the dependent variable is the difference between log wages in 2000 and 1995 in county i, and the explanatory variable is Internet investment by businesses in county i in 2000. This regression corresponds to Regression 2. Table 4 presents regressions where advanced internet investment is instrumented by a county level variable. This regression is the first-difference version of Regression 3. Because the data only bears two periods, these two regressions are algebraically equivalent.

In regression equation (1), the dependent variable is the price of houses in region r at time t, while the explanatory variables include region and time fixed effects, and the number of people killed because of the civil war in region r at time t-1. This regression corresponds to Regression 1.


34
In regression equation (1), the dependent variable is the change of the log premium for employer e in market m in year t, and explanatory variables are time and market fixed effects, and the change in various treatment variables (change in the fraction of self-insured employees...). This regression is similar to Regression 2, except that it has several treatment variables, and market and time fixed effects.

27. Hornbeck (2012), The Enduring Impact of the American Dust Bowl: Short-and Long-Run Adjustments to Environmental Catastrophe. *Table 2.* In regression equation (1), the dependent variable is, say, the change in log land value in county c between period t and 1930, and the explanatory variables are state × year effects, the share of county c in high erosion, and the share of county c in medium erosion. This regression is similar to Regression 1, except that it has two treatment variables and state-year fixed effects.


In, say, the first regression equation in the bottom of page 1915, the dependent variable is the change in the price of house j between sales 2 and 3, and the explanatory variables are the change in various pollutants in the area around house j between sales 2 and 3. This regression is similar to Regression 2, except that it has several treatment variables.


In the reduced form of regression equation (4), the dependent variable is the change in test scores for child i between years a and a-1, while the explanatory variable is the change in the expected EITC income of her family based on her family income in year a-1. This regression corresponds to Regression 2. The first stage is the same regression but with the change in the income of the family of student i between years a and a-1. Overall, the 2SLS coefficient arising from regression equation (4) is a ratio of 2 weighted averages of Wald-DIDs.


In regression equation (1), the dependent variable is the test score of student i in school j in grade g and year t, and the explanatory variables are grade, school, year, and grade × year effects, and the fraction of Katrina students received by school j in grade g and year t. Within each grade, this regression corresponds to Regression 1.

In regression equation (1), the dependent variable is the value of investment in firm i and year t divided by the lagged book value of properties, plants, and equipments (PPE), and the explanatory variables are firm and time fixed effects and the market value of firm i in year t divided by its lagged PPE. This regression corresponds to Regression 1.

In regression equation (1), the outcome variable is, say, income of household i at period t, and the explanatory variables include household and time fixed effects, and the minimum wage in the state where household i lives in period t. This regression corresponds to Regression 1.

In regression equation (7), the dependent variable is a measure of skills in the labor force employed by company i in industry j at period t, and the explanatory variables are firm and industry \times period fixed effects, the ratio of exports to sales in firm i at period t, and the share of firm exports to high income destinations over total exports. To instrument this variable, the authors use an indicator for the years 1999 or 2000 (a large devaluation happened in Brazil in 1999) interacted with the share of exports of firm i to Brazil in 1998. This regression corresponds to Regression 3.

34. Faye and Niehaus (2012), Political Aid Cycles. Table 3, columns 4 and 5, and Tables 4 and 5.
In regression equation (2), the dependent variable is the amount of donations received by receiver r from donor d in year t, and the explanatory variables are donor \times receiver fixed effects, an indicator for whether there is an election in country r in year t, a measure of alignment between the ruling political parties in countries r and d, and the interaction of the election indicator and the measure of alignment. This regression corresponds to Regression 1.

B Proofs

B.1 A useful lemma

For all \((g, g', t, t') \in \{0, ..., g\}^2 \times \{1, ..., t\}^2\), let

\[
DID_R(g, g', t, t') = E(R_{g,t}) - E(R_{g,t'}) - (E(R_{g',t}) - E(R_{g',t'})).
\]

Our lemma relates the \(DID_Y(g, g', t, t')\) estimands to the \(ACR^{TR}_{g,t}\) and \(ACR^S_{g,t}\) parameters.
Lemma 1 Assume that $D \in \{0, \ldots, \overline{d}\}$.

1. If Assumption 1 holds, for all $(g, g', t, t') \in \{0, \ldots, \overline{g}\} \times \{1, \ldots, \overline{t}\}^2$

$$\text{DID}_Y(g, g', t, t') = E(D_{g,t}) \text{ACR}^{TR}_{g,t} - E(D_{g',t'}) \text{ACR}^{TR}_{g',t'} - (E(D_{g,t}) \text{ACR}^{TR}_{g',t} - E(D_{g',t}) \text{ACR}^{TR}_{g',t'}).$$

2. If Assumptions 1, 2, and 3O hold, for all $(g, g', t, t') \in \{0, \ldots, \overline{g}\} \times \{1, \ldots, \overline{t}\}^2$

$$\text{DID}_Y(g, g', t, t - 1) = (E(D_{g,t}) - E(D_{g,t-1})) \text{ACR}^S_{g,t} - (E(D_{g',t}) - E(D_{g',t-1})) \text{ACR}^S_{g',t}.$$

In the special case where $D$ is binary, Lemma 1 can be rewritten as follows. The second statement of Corollary 3 is close to Theorem 1 in de Chaisemartin and D’Haultfœuille (2017a).

Corollary 3 Assume that $D$ is binary.

1. If Assumption 1 holds, for all $(g, g', t, t') \in \{0, \ldots, \overline{g}\} \times \{1, \ldots, \overline{t}\}^2$

$$\text{DID}_Y(g, g', t, t') = E(D_{g,t}) \Delta^{TR}_{g,t} - E(D_{g',t'}) \Delta^{TR}_{g',t'} - (E(D_{g,t}) \Delta^{TR}_{g',t} - E(D_{g',t}) \Delta^{TR}_{g',t'}).$$

2. If Assumptions 1-3 hold, for all $(g, g', t, t') \in \{0, \ldots, \overline{g}\} \times \{1, \ldots, \overline{t}\}^2$

$$\text{DID}_Y(g, g', t, t - 1) = (E(D_{g,t}) - E(D_{g,t-1})) \Delta^S_{g,t} - (E(D_{g',t}) - E(D_{g',t-1})) \Delta^S_{g',t}.$$

The first statement of the corollary follows from the first statement of Lemma 1, once noted that when $D$ is binary, $\text{ACR}^{TR}_{g,t} = \Delta^{TR}_{g,t}$ for all $(g, t) \in \{0, \ldots, \overline{g}\} \times \{1, \ldots, \overline{t}\}$. The second statement follows from the second statement of Lemma 1, once noted that when $D$ is binary, Assumptions 3 and 3O are the same, and $\text{ACR}^S_{g,t} = \Delta^S_{g,t}$ for all $(g, t) \in \{0, \ldots, \overline{g}\} \times \{1, \ldots, \overline{t}\}$.

Proof of Lemma 1

1. We have

$$\text{DID}_Y(g, g', t, t') = E(Y_{g,t}) - E(Y_{g',t'}) - (E(Y_{g',t}) - E(Y_{g',t'})). \quad (7)$$

Moreover,

$$E(Y_{g,t}) = E(Y_{g,t}(0)) + E[Y_{g,t}(D) - Y_{g,t}(0)]. \quad (8)$$

The result follows by decomposing similarly the three other terms of $\text{DID}_Y(g, g', t, t')$, plugging these decompositions into (7), using Assumption 1, and finally using the definition of $\text{ACR}^{TR}_{g,t}$.

2. We prove the result when $E(D_{g,t}) \geq E(D_{g,t-1})$ and $E(D_{g',t}) \geq E(D_{g',t-1})$. The proof is similar in the three other cases. First,

$$E(Y_{g,t-1})) = E(Y_{g,t-1}(0)) + \sum_{d=1}^{\overline{d}} P(D_{g,t-1}(t - 1) = d) E(Y_{g,t-1}(d) - Y_{g,t-1}(0)|D(t - 1) = d)$$

$$= E(Y_{g,t-1}(0)) + \sum_{d=1}^{\overline{d}} P(D_{g,t}(t - 1) = d) E(Y_{g,t}(d) - Y_{g,t}(0)|D(t - 1) = d). \quad (9)$$

37
where the second equality follows from Assumptions 2 and 3O. Similarly,

\[ E(Y_{g,t}) = E(Y_{g,t}(0)) + \sum_{d=1}^{\bar{d}} P(D_{g,t}(t) = d) E(Y_{g,t}(d) - Y_{g,t}(0)|D(t) = d). \]  \hspace{1cm} (10)

Combining Equations (9) and (10) yields

\[ E(Y_{g,t}) - E(Y_{g,t-1}) = E(Y_{g,t}(0)) - E(Y_{g,t-1}(0)) \]

\[ + E \left[ \sum_{d=1}^{\bar{d}} (Y_{g,t}(d) - Y_{g,t}(0)) \{1 \{D_{g,t}(t) = d\} - 1 \{D_{g,t}(t - 1) = d\}\} \right]. \]  \hspace{1cm} (11)

Now, remark that

\[ \sum_{d=1}^{\bar{d}} (Y_{g,t}(d) - Y_{g,t}(0)) \{1 \{D_{g,t}(t) = d\} - 1 \{D_{g,t}(t - 1) = d\}\} \]

\[ = \sum_{d=1}^{\bar{d}} (Y_{g,t}(d) - Y_{g,t}(d - 1)) \{1 \{D_{g,t}(t) \geq d\} - 1 \{D_{g,t}(t - 1) \geq d\}\} \]

\[ = \sum_{d=1}^{\bar{d}} (Y_{g,t}(d) - Y_{g,t}(d - 1)) 1 \{D_{g,t}(t) \geq d > D_{g,t}(t - 1)\}. \] \hspace{1cm} (12)

The first equality follows by summation by parts. The second uses the fact that under Assumption 2, \( E(D_{g,t}) \geq E(D_{g,t-1}) \) implies that \( D_{g,t}(t) \geq D_{g,t}(t - 1) \). Then,

\[ E(Y_{g,t}) - E(Y_{g,t-1}) - (E(Y_{g,t}(0)) - E(Y_{g,t-1}(0))) \]

\[ = \sum_{d=1}^{\bar{d}} P(D_{g,t}(t) \geq d > D_{g,t}(t - 1)) E(Y_{g,t}(d) - Y_{g,t}(d - 1)|D_{g,t}(t) \geq d > D_{g,t}(t - 1)) \]

\[ = \sum_{d=1}^{\bar{d}} [P(D_{g,t} \geq d) - P(D_{g,t-1} \geq d)] E(Y_{g,t}(d) - Y_{g,t}(d - 1)|D_{g,t}(t) \geq d > D_{g,t}(t - 1)) \]

\[ = (E(D_{g,t}) - E(D_{g,t-1}))ACR_{g,t}^S. \] \hspace{1cm} (13)

The first equality follows from plugging Equation (12) into Equation (11). The second uses the fact that under Assumption 2, \( E(D_{g,t}) \geq E(D_{g,t-1}) \) implies that \( D_{g,t}(t) \geq D_{g,t}(t - 1) \).

One can follow the same steps to show that

\[ E(Y_{g',t}) - E(Y_{g',t-1}) - (E(Y_{g',t}(0)) - E(Y_{g',t-1}(0))) = (E(D_{g',t}) - E(D_{g',t-1}))ACR_{g',t}^S. \] \hspace{1cm} (14)

Finally, the result follows by taking the difference between Equations (13) and (14), and using Assumption 1.
B.2 Proof of Theorem 1

Proof of the first statement.

Result for $k = 1$.

It follows from the Frisch-Waugh theorem and the definition of $\varepsilon_{1,G,T}$ that

$$\beta_1 = \frac{\text{cov}(\varepsilon_{1,G,T}, Y)}{\text{cov}(\varepsilon_{1,G,T}, E(D|G,T))}. \quad (15)$$

Note that the denominator must be different from 0, otherwise $\beta_1$ would not be well defined. As a result,

$$E(\varepsilon_{1,G,T}E(D|G,T)) = \text{cov}(\varepsilon_{1,G,T}, E(D|G,T)) \neq 0. \quad (16)$$

Now, by definition of $\varepsilon_{1,G,T}$ again,

$$E[\varepsilon_{1,G,T}|G] = 0, \quad E[\varepsilon_{1,G,T}|T] = 0. \quad (17)$$

Then,

$$\text{cov}(\varepsilon_{1,G,T}, Y) = E[\varepsilon_{1,G,T}E(Y|G,T)]
= E[\varepsilon_{1,G,T}DID_Y(G,0,T,0)]
= E[\varepsilon_{1,G,T}(E(D|G,T)\Delta^{TR}_{G,T} - E(D_{G,0})\Delta^{TR}_{G,0} - E(D_{0,T})\Delta^{TR}_{0,T} + E(D_{0,0})\Delta^{TR}_{0,0})]
= E[\varepsilon_{1,G,T}E(D|G,T)\Delta^{TR}_{G,T}]
= E[\varepsilon_{1,G,T}D\Delta^{TR}_{G,T}]. \quad (18)$$

The first equality follows from the law of iterated expectations. (17) implies that

$$E[\varepsilon_{1,G,T}(-E(Y_{G,0}) - E(Y_{0,T}) + E(Y_{0,0}))] = 0,$$

hence the second equality. The third equality follows from the first point of Corollary 3. (17) implies that

$$E[\varepsilon_{1,G,T}(-E(D_{G,0})\Delta^{TR}_{G,0} - E(D_{0,T})\Delta^{TR}_{0,T} + E(D_{0,0})\Delta^{TR}_{0,0})] = 0,$$

hence the fourth equality. The fifth equality follows from the law of iterated expectations. Combining (15), (16), and (18), we obtain

$$\beta_1 = \frac{E[\varepsilon_{1,G,T}D\Delta^{TR}_{G,T}]}{E(\varepsilon_{1,G,T}E(D|G,T))}
= \frac{E[W_1D\Delta^{TR}_{G,T}]}{E(D)}
= E[W_1\Delta^{TR}_{G,T}|D = 1],$$
where the second equality follows from the definition of \( W_1 \).

**Result for** \( k = 2 \).

With a slight abuse of notation, let, \( E(Y_{g,t}), \Delta_{g,T}^{TR}, \) and \( E(D_{g,t}) \) respectively denote \( E(Y_{g,t}), \Delta_{g,t}^{TR}, \) and \( E(D_{g,t}) \) evaluated at \((g,t) = (G,T)\).

It follows from Frisch-Waugh theorem and the definition of \( \varepsilon_{2,G,T} \) that

\[
\beta_2 = \frac{\text{cov}(\varepsilon_{2,G,T}, FD_Y(G,T)|T \geq 1)}{\text{cov}(\varepsilon_{2,G,T}, FD_D(G,T)|T \geq 1)}.
\]  

(19)

Note that the denominator must be different from 0, otherwise \( \beta_2 \) would not be well defined.

Now, by definition of \( \varepsilon_{2,G,T} \) again,

\[
E[\varepsilon_{2,G,T}|T] = 0.
\]  

(20)

Then,

\[
\text{cov}(\varepsilon_{2,G,T}, FD_Y(G,T)|T \geq 1)
= E[\varepsilon_{2,G,T}DID_Y(G,0,T,T-1)|T \geq 1]
= E \left[ \varepsilon_{2,G,T} \left( E(D_{G,T})\Delta_{g,T}^{TR} - E(D_{G,T-1})\Delta_{g,T-1}^{TR} - E(D_{0,T})\Delta_{0,T}^{TR} + E(D_{0,T-1})\Delta_{0,T-1}^{TR} \right) |T \geq 1 \right]
= E \left[ \varepsilon_{2,G,T} \left( E(D_{G,T})\Delta_{g,T}^{TR} - E(D_{G,T-1})\Delta_{g,T-1}^{TR} \right) |T \geq 1 \right]
= \sum_{g=0}^{\bar{g}} \sum_{t=0}^{\bar{t}} E(\varepsilon_{2,G,T}1\{G = g\}1\{T = t\}|T \geq 1) \left( E(D_{g,t})\Delta_{g,t}^{TR} - E(D_{g,t-1})\Delta_{g,t-1}^{TR} \right)
= \sum_{g=0}^{\bar{g}} \sum_{t=0}^{\bar{t}} \varepsilon_{2,g,t} \left( \frac{P(G = g,T = t)}{P(T \geq 1)} - \frac{P(G = g,T = t + 1)}{P(T \geq 1)} \right) E(D_{g,t})\Delta_{g,t}^{TR}
= \frac{1}{P(T \geq 1)} \sum_{g=0}^{\bar{g}} \sum_{t=0}^{\bar{t}} P(G = g,T = t)\tilde{\omega}_{2,g,t} E(D_{g,t})\Delta_{g,t}^{TR}
= \frac{1}{P(T \geq 1)} E \left[ \tilde{\omega}_{2,G,T}D\Delta_{G,T}^{TR} \right].
\]

Eq. (20) implies that

\[
E[\varepsilon_{2,G,T} - (E(Y_{0,T}) - E(Y_{0,T-1}))[T \geq 1] = 0,
\]

hence the first equality. The second equality follows from the first point of Corollary 3. Eq. (20) implies again that

\[
E[\varepsilon_{2,G,T} - (E(D_{0,T})\Delta_{0,T}^{TR} - E(D_{0,T-1})\Delta_{0,T-1}^{TR})|T \geq 1] = 0,
\]

40
hence the third equality. The fifth equality follows from a summation by part, the sixth holds because \( \varepsilon_{2,g,0} = 0 \), the seventh follows from the definition of \( \tilde{\omega}_{2,g,t} \), and the eighth from the law of iterated expectations.

A similar reasoning yields

\[
\text{cov}(\varepsilon_{2,G,T}, FD_D(G, T) | T \geq 1) = \frac{1}{P(T \geq 1)} E[\tilde{\omega}_{2,G,T} D].
\]

The end of the proof follows exactly as for \( k = 1 \).

Result for \( k = 3 \).

It follows from the Frisch-Waugh theorem and the definition of \( \varepsilon_{3,G,T} \) that

\[
\beta_3 = \frac{\text{cov}(\varepsilon_{3,G,T}, Y)}{\text{cov}(\varepsilon_{3,G,T}, D)}.
\]

Then, notice that

\[
\text{cov}(\varepsilon_{3,G,T}, D) = E(\varepsilon_{3,G,T} D) = E(\varepsilon_{3,G,T} E(D|G, T)) = \text{cov}(\varepsilon_{3,G,T}, E(D|G, T)).
\]

Therefore,

\[
\beta_3 = \frac{\text{cov}(\varepsilon_{3,G,T}, Y)}{\text{cov}(\varepsilon_{3,G,T}, E(D|G, T))}.
\]

The end of the proof follows exactly as for \( k = 1 \).

Proof of the second statement.

We first prove Equation (1) and two other equalities that we use in the proof. Note first that for all \((g, t) \in \{0, \ldots, \bar{g}\} \times \{1, \ldots, \bar{t}\}\), we have

\[
|E(D_{g,t}) - E(D_{g,t-1})| = |P(D(t) = 1|G = g, T = t) - P(D(t - 1) = 1|G = g, T = t - 1)| = |P(D(t) = 1|G = g, T = t) - P(D(t - 1) = 1|G = g, T = t)| = P(S|G = g, T = t),
\]

where the equalities follow from Assumption 2. Then,

\[
E[f(G, T)1_S] = E[f(G, T)P(S|G, T)] = \sum_{g=0}^{\bar{g}} \sum_{t=1}^{\bar{t}} P(G = g, T = t) |E(D_{g,t}) - E(D_{g,t-1})| f(g, t).
\]

The first equality follows from the law of iterated expectations, and the second follows from the fact that \( P(S|G = g, T = 0) = 0 \) and from the previous display. Similarly,

\[
P(S) = \sum_{g=0}^{\bar{g}} \sum_{t=1}^{\bar{t}} P(G = g, T = t) |E(D_{g,t}) - E(D_{g,t-1})|.
\]
(1) follows by dividing (23) by (24).

Proof for $k = 1$.

We have

\[
\text{cov}(\varepsilon_{1,G,T}, Y) = E[\varepsilon_{1,G,T}(E(Y|G, T) - E(Y|G = 0, T))] \\
= E \left( \sum_{t=0}^{\tau} E(\varepsilon_{1,G,T}1\{T = t\}|G)[E(Y|G, T = t) - E(Y|G = 0, T = t)] \right) \\
= E \left( \sum_{t=1}^{\tau} \left[ \sum_{t' \geq t} E(\varepsilon_{1,G,T}1\{T = t'\}|G) \right] D I D_{Y}(G, 0, t, t - 1) \right) \\
= E \left( \sum_{t=1}^{\tau} E(\varepsilon_{1,G,T}1\{T \geq t\}|G) \left[ FD_{D}(G, t)\Delta_{G,t}^S - FD_{D}(0, t)\Delta_{0,t}^S \right] \right) \\
= E \left( \sum_{t=1}^{\tau} E(\varepsilon_{1,G,T}1\{T \geq t\}|G)FD_{D}(G, t)\Delta_{G,t}^S \right) - \sum_{t=1}^{\tau} E(\varepsilon_{1,G,T}1\{T \geq t\})FD_{D}(0, t)\Delta_{0,t}^S \\
= \sum_{g=0}^{\eta} \sum_{t=1}^{\tau} P(G = g, T = t)|FD_{D}(g, t)| \frac{s_{g,t}E[\varepsilon_{1,G,T}1\{G = g, T \geq t\}]}{P(G = g, T = t)} \Delta_{g,t} \\
= E \left[ \tilde{\omega}_{1,G,T}\Delta_{G,T}^S 1_S \right]. \tag{25}
\]

The first equality follows from the law of iterated expectation and (17). The third equality follows from summation by part and (17). The fourth equality follows from the second point of Corollary 3. (25) follows from (17). (26) follows from the definition of $\tilde{\omega}_{1,g,t}$ and from (23).

Similarly,

\[
\text{cov}(\varepsilon_{1,G,T}, E(D|G, T)) = \sum_{g=0}^{\eta} \sum_{t=1}^{\tau} P(G = g, T = t)|FD_{D}(g, t)| \frac{s_{g,t}E[\varepsilon_{1,G,T}1\{G = g, T \geq t\}]}{P(G = g, T = t)} \Delta_{g,t} \tag{27}
\]

\[
= E \left[ \tilde{\omega}_{1,G,T} 1_S \right]. \tag{28}
\]

Combining (15), (26), and (28) yields

\[
\beta_1 = \frac{E \left[ \tilde{\omega}_{1,G,T}\Delta_{G,T}^S 1_S \right]}{E \left[ \tilde{\omega}_{1,G,T} 1_S \right]}.
\]

Finally, the result follows from the definition of $\Omega_1$, after dividing the numerator and the denominator in the rhs of the previous display by $P(S)$.

Proof for $k = 2$. 

42
First,
\[
\text{cov}(\varepsilon_{2,G,T}, FD_Y(G,T)|T \geq 1) = E[\varepsilon_{2,G,T} D I D_Y(G,0,T,T-1)|T \geq 1]
\]
\[
= E[\varepsilon_{2,G,T} (FD_D(G,T) \Delta^S_{G,T} - FD_D(0,T) \Delta^S_{0,T}) |T \geq 1]
\]
\[
= E[\varepsilon_{2,G,T} FD_D(G,T) \Delta^S_{G,T} |T \geq 1]
\]
\[
= E[\varepsilon_{2,G,T} s_{G,T} |FD_D(G,T)| \Delta^S_{G,T} |T \geq 1]
\]
\[
= E[\varepsilon_{2,G,T} s_{G,T} \Delta^S_{G,T} 1_s]
\]
\[
= \frac{E[\varepsilon_{2,G,T} s_{G,T} \Delta^S_{G,T} 1_s]}{P(T \geq 1)}.
\]

The first equality follows from (20). The second equality follows from the second point of Corollary 3. The third equality follows from (20). The fifth equality follows from (23).

Similarly,
\[
\text{cov}(\varepsilon_{2,G,T}, FD_D(G,T)|T \geq 1) = \frac{E[\varepsilon_{2,G,T} s_{G,T} 1_s]}{P(T \geq 1)}.
\]

The result follows combining (19) and the two previous displays.

**Proof for \( k = 3 \).**

Following the exact same steps as those used to prove (25) and (27), one can show that
\[
\text{cov}(\varepsilon_{3,G,T}, Y) = \sum_{g=0}^{\bar{g}} \sum_{t=1}^{\bar{t}} P(G = g, T = t)|FD_D(g,t)| \frac{s_{g,t} E[\varepsilon_{3,G,T} 1\{G = g, T \geq t\}]}{P(G = g, T = t)} \Delta^S_{g,t}
\]
\[
\text{cov}(\varepsilon_{3,G,T}, D) = \sum_{g=0}^{\bar{g}} \sum_{t=1}^{\bar{t}} P(G = g, T = t)|FD_D(g,t)| \frac{s_{g,t} E[\varepsilon_{3,G,T} 1\{G = g, T \geq t\}]}{P(G = g, T = t)}
\]

Now, because \( \bar{t} = 1 \), we actually we have
\[
\text{cov}(\varepsilon_{3,G,T}, Y) = \sum_{g=0}^{\bar{g}} P(G = g, T = 1)|FD_D(g,1)| \varepsilon_{3,g,1} \Delta^S_{g,1}
\]
\[
\text{cov}(\varepsilon_{3,G,T}, D) = \sum_{g=0}^{\bar{g}} P(G = g, T = 1)|FD_D(g,1)| \varepsilon_{3,g,1}.
\]

Finally, the result follows after combining (21), (23), and the previous display.

**B.3 Proof of Corollary 2**

We prove only the first statement, as the second statement can be proven by following the exact same steps. If the assumptions of the corollary hold and if \( \Delta^{TR} = 0 \), then
\[
\begin{align*}
\beta_k &= E \left[ W_k \Delta^{TR}_{G,T} | D = 1 \right], \\
0 &= E \left[ \Delta^{TR}_{G,T} | D = 1 \right].
\end{align*}
\]

43
These two conditions and the Cauchy-Schwarz inequality imply

\[ |\beta_k| = \left| \text{cov} \left( W_k, \Delta_{G,T}^{TR} \bigg| D = 1 \right) \right| \leq V(W_k|D = 1)^{1/2} V(\Delta_{G,T}^{TR}|D = 1)^{1/2}. \]

Hence, \( \sigma^{TR} \geq \sigma_k^{TR} \).

Now, we prove that we can rationalize this lower bound. Let us define

\[ \Delta_{G,T}^{TR} = \frac{\beta_k (W_k - 1)}{V(W_k|D = 1)}. \]

Then

\[ \Delta^{TR} = E[\Delta_{G,T}^{TR}|D = 1] = \frac{\beta_k}{V(W_k|D = 1)} E[W_k - 1|D = 1] = 0. \]

Similarly,

\[ E[W_k \Delta_{G,T}^{TR}|D = 1] = \frac{\beta_k}{V(W_k|D = 1)} E[W_k^2 - W_k|D = 1] \]
\[ = \frac{\beta_k}{V(W_k|D = 1)} V(W_k|D = 1) \]
\[ = \beta_k. \]

This proves the result.

B.4 Proof of Proposition 1

Proof of the first statement.

Let \( I_{g,t} = 1\{E(D_{g,t}) \neq 0 \} \). We reason by contradiction, by showing that if one of the two conditions does not hold and if

\[ w_{k,g,t}I_{g,t} = I_{g,t} \forall (g,t), \tag{29} \]

which is equivalent to \( V(W_k|D = 1) = 0 \), then the denominator \( \text{Den}(w_k) \) of \( w_{k,g,t} \) is equal to 0. This contradicts the fact that \( \beta_k \) is well-defined.

When \( k = 1 \), (29) implies that for all \((g,t)\), there exists a constant \( C \) such that \( \varepsilon_{1,g,t}I_{g,t} = CI_{g,t} \). First suppose that there exists \( g_0 \) such that for all \( t \), \( I_{g_0,t} = 1 \). Then

\[ C = E[CI_{g_0,T}|G = g_0] \]
\[ = E[\varepsilon_{1,g_0,T}I_{g_0,T}|G = g_0] \]
\[ = CE[\varepsilon_{1,G,T}|G = g_0] \]
\[ = 0, \]
where the last equality follows by (17). Alternatively, suppose that there exists \( t_0 \) such that for all \( g, I_{g,t_0} = 1 \). By taking expectation over \( G \) instead of \( T \) and using (17) again, we also obtain \( C = 0 \). Then, in both cases,

\[
\text{Den}(w_k) = E[E(D(G,T)\varepsilon_{1,G,T})] \\
= E[E(D(G,T)I_{G,T}\varepsilon_{1,G,T})] \\
= E[E(D(G,T)CI_{G,T}] \\
= 0.
\]

When \( k = 2 \), (29) implies that for all \((g,t)\), there exists a constant \( C \) such that \( \tilde{w}_{2,g,t}I_{g,t} = CI_{g,t} \). Then, if there exists \( g_0 \) such that for all \( t, I_{g_0,t} = 1 \), we get

\[
C = E[CI_{g_0,T}|G = g_0] \\
= E[\tilde{w}_{2,g_0,T}|G = g_0] \\
= \frac{1}{P(G = g_0)} \sum_{t=0}^{t} (P(G = g_0, T = t)\varepsilon_{2,g_0,t} - P(G = g_0, T = t + 1)\varepsilon_{2,g_0,t+1}) \\
= \frac{1}{P(G = g_0)} (P(G = g_0, T = \tilde{t} + 1)\varepsilon_{2,g_0,0} - P(G = g_0, T = 0)\varepsilon_{2,g_0,\tilde{t}+1}) \\
= 0.
\]

If there exists \( t_0 \) such that for all \( g, I_{g,t_0} = 1 \), we obtain similarly

\[
C = E[CI_{g,t_0}|T = t_0] \\
= E[\tilde{w}_{2,g,t_0}|T = t_0] \\
= \frac{1}{P(T = t_0)} \sum_{g=0}^{g} (P(G = g, T = t_0)\varepsilon_{2,g,t_0} - P(G = g, T = t_0 + 1)\varepsilon_{2,g,t_0+1}) \\
= E[\varepsilon_{2,G,T}|T = t_0] - \frac{P(T = t_0 + 1)}{P(T = t_0)} E[\varepsilon_{2,G,T}|T = t_0 + 1] \\
= 0,
\]

where the last equality follows from (20). In both cases, one can then use the same argument as for \( k = 1 \) to show that the denominator of \( w_{2,g,t} \) is equal to 0.

Finally, when \( k = 3 \), the result follows by applying the exact same reasoning as when \( k = 1 \).

**Proof of the second statement.**

Let \( I'_{g,t} = 1 \{E(D_{g,t}) \neq E(D_{g,t-1}) \} \), so that \( P(S|G = g, T = t) > 0 \) if and only if \( I'_{g,t} = 1 \). As above, we reason by contradiction by showing that if the one of the conditions does not hold and if

\[
\omega_{k,g,t}I'_{g,t} = I'_{g,t}, \forall (g,t),
\]

(30)
which is equivalent to $V(\Omega_k|S) = 0$, then the denominator $\text{Den}(\omega_k)$ of $\omega_{k,g,t}$ is equal to 0.

When $k = 1$, (30) together with the definition of $\omega_{1,g,t}$ and the equalities $s_{g,t}I'_{g,t} = s_{g,t}$ and $s_{g,t}^2I'_{g,t} = I'_{g,t}$ imply that for some constant $C$ and all $(g, t)$,

$$E(\varepsilon_{1,G,T} 1\{G = g, T \geq t\})I'_{g,t} = CP(G = g, T = t)s_{g,t}.$$  

Suppose that there exists $t_0$ such that $g \mapsto s_{g,t_0} = s$, with $s \in \{-1, 1\}$. Then

$$CP(T = t_0)s = \sum_{g=0}^{\bar{g}} CP(G = g, T = t_0)s_{g,t_0} = \sum_{g=0}^{\bar{g}} E(\varepsilon_{1,G,T} 1\{G = g, T \geq t_0\}) = E(\varepsilon_{1,G,T} 1\{T \geq t_0\}) = 0$$

where the last equality follows by (17). One can then use the same reasoning as that used in the proof of the first statement to show that the previous equality implies that $\text{Den}(\omega_1) = 0$.

When $k = 2$, (30) implies that for some $C$ and all $(g, t)$, $\varepsilon_{2,G,T}I'_{G,T} = Cs_{G,T}$. Suppose that there exists $t_0$ such that $g \mapsto s_{g,t_0} = s$, where $s \in \{-1, 1\}$. Then $Cs = E(\varepsilon_{2,G,T}|T = t_0)$, which yields $C = 0$ by (20). Again, this implies that $\text{Den}(\omega_2) = 0$.

Finally, when $k = 3$, the result follows by applying the exact same reasoning as with $k = 2$.

**B.5 Proof of Proposition 2**

1. We have

$$P(W_1 \geq 0|D = 1) = \sum_{g=0}^{\bar{g}} \sum_{t=0}^{\bar{t}} 1\{w_{1,g,t} \geq 0\} P(G = g, T = t|D = 1).$$

Therefore, $P(W_1 \geq 0|D = 1) = 1$ is equivalent to having that $w_{1,g,t}P(D = 1|G = g, T = t) \geq 0$ for all $(g, t) \in \{0, ..., \bar{g}\} \times \{0, ..., \bar{t}\}$. A few lines of algebra show that $E(\varepsilon_{1,G,T}E(D|G, T)) = V(\varepsilon_{1,G,T})$, which is strictly positive if $\beta_1$ is well-defined. Then, to study the sign of $w_{1,g,t}P(D = 1|G = g, T = t)$, it is sufficient to study the sign of $\varepsilon_{1,g,t}1\{E(D_{g,t}) > 0\}$. Under Assumption 7, $G \perp \perp T$. If $G \perp \perp T$, one can show that

$$\varepsilon_{1,g,t} = E(D_{g,t}) - E(D_{g,\cdot}) - E(D_{\cdot,t}) + E(D).$$
Then, Assumptions 7 and 6 imply that for all \((g, t) \in \{0, \ldots, \bar{g}\} \times \{0, \ldots, \bar{t}\},\)
\[
\varepsilon_{1,g,t}1\{E(D_{g,t}) > 0\} = 1\{t \geq a_g\} \left(1 - \frac{\bar{t} + 1 - a_g}{\bar{t} + 1} - \frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} \leq t\} + \frac{1}{\bar{t} + 1} \sum_{t' = 0}^{\bar{t}} \frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} \leq t'\}\right).
\]
\[
\varepsilon_{1,g,t}1\{E(D_{g,t}) > 0\} = 0
\]
for all \((g, t)\) such that \(t < a_g\). For all \((g, t)\) such that \(t \geq a_g\), \(t \mapsto \varepsilon_{1,g,t}1\{E(D_{g,t}) > 0\}\) is decreasing in \(t\). Moreover, \(g \mapsto \varepsilon_{1,g,t}\) is minimized at \(g = g(e)\). Therefore, \(\varepsilon_{1,g,t}1\{E(D_{g,t}) > 0\}\) is strictly negative for some \((g, t)\) if and only if
\[
\frac{a_{g(e)}}{\bar{t} + 1} - \frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} \leq \bar{t}\} + \frac{1}{\bar{t} + 1} \sum_{t' = 0}^{\bar{t}} \frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} \leq t'\} < 0.
\]
The proof is similar for \(\Omega_1\) and is therefore omitted.

2. Here as well, \(P(W_2 \geq 0|D = 1) = 1\) is equivalent to having that \(w_{2,g,t}P(D = 1|G = g, T = t) \geq 0\) for all \((g, t) \in \{0, \ldots, \bar{g}\} \times \{0, \ldots, \bar{t}\}\). Moreover, under the assumptions of the theorem, to study the sign of \(w_{2,g,t}P(D = 1|G = g, T = t)\), it is sufficient to study the sign of \(v_{g,t} = \bar{w}_{2,g,t}1\{E(D_{g,t}) > 0\}\). For all \((g, t) \in \{0, \ldots, \bar{g}\} \times \{0, \ldots, \bar{t}\},\)
\[
v_{g,t} = 1\{t \geq a_g\}(\varepsilon_{2,g,t} - \varepsilon_{2,g,t+1})
\]
\[
= 1\{t \geq a_g\}[1\{t \geq 1\}(E(D_{g,t}) - E(D_{g,t-1}) - E(D_{t,t-1})) - 1\{t \leq \bar{t} - 1\}(E(D_{g,t+1}) - E(D_{g,t}) - E(D_{t,t+1}) + E(D_{t,t}))]
\]
\[
= 1\{t \geq a_g\}[1\{t \geq 1\}1\{t = a_g\} - \frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} = t\} + \frac{1}{\bar{t} + 1} \sum_{t' = 0}^{\bar{t}} \frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} = t + 1\}].
\]
The first equality follows from the definition of \(\bar{w}_{2,g,t}\), and from Assumptions 7 and 6. The second equality follows from the fact that for all \((g, t) \in \{0, \ldots, \bar{g}\} \times \{0, \ldots, \bar{t} + 1\},\)
\[
\varepsilon_{2,g,t} = 1\{1 \leq t \leq \bar{t}\}(E(D_{g,t}) - E(D_{g,t-1}) - E(D_{t,t}) + E(D_{t,t-1})).
\]
The third equality follows from Assumptions 7 and 6.

If \(a_g \geq \bar{t}\) for every \(g \in \{0, \ldots, \bar{g}\}\), it follows from the previous display that \(v_{g,t} \geq 0\) for every \((g, t) \in \{0, \ldots, \bar{g}\} \times \{0, \ldots, \bar{t}\}\). Conversely, assume that at least one group adopts before \(\bar{t}\). Then, \(a_{g(e)} = t_0 < \bar{t}\). Now, we reason by contradiction. Assume that \(v_{g(e),t_0+1}, \ldots, v_{g(e),\bar{t}} \geq 0\). \(v_{g(e),\bar{t}} \geq 0\) implies \(\frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} = \bar{t}\} = 0\). Then, \(v_{g(e),\bar{t}-1} \geq 0\) implies \(\frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} = \bar{t} - 1\} = 0\). And so on and so forth. Finally, \(v_{g(e),t_0+1} \geq 0\) implies \(\frac{1}{\bar{g} + 1} \sum_{g' = 0}^{\bar{g}} 1\{a_{g'} = t_0 + 1\} = 0\). Therefore, all groups must have \(a_g = a_{g(e)}\). But then, \(E(D_{g,t}) - E(D_{g,t-1}) = 1\{t = a_{g(e)}\}\) if \(a_{g(e)} \geq 1\), and
\[
E(D_{g,t}) - E(D_{g,t-1}) = 0 \text{ otherwise. This contradicts the fact that } \beta_2 \text{ is well-defined. Therefore, at least one of the } (\upsilon_{g(\epsilon),t_0+1}, \ldots, \upsilon_{g(\epsilon),t}) \text{ must be strictly negative.}
\]

3. Here as well, \( P(\Omega_2 \geq 0|S) = 1 \) is equivalent to having that \( \omega_{2,g,t} P(S|G = g, T = t) \geq 0 \) for all \( (g, t) \in \{0, \ldots, \bar{g}\} \times \{1, \ldots, \bar{t}\} \). Moreover, to study the sign of \( \omega_{2,g,t} P(S|G = g, T = t) \), it is sufficient to study the sign of \( s_{g,t} \epsilon_{2,g,t} P(S|G = g, T = t) \). For all \( (g, t) \in \{0, \ldots, \bar{g}\} \times \{1, \ldots, \bar{t}\} \),

\[
\begin{align*}
&= s_{g,t} [E(D_{g,t}) - E(D_{g,t-1}) - E(D_{.:t}) + E(D_{.:t-1})] \\
&= 1 \{t = a_g\} [1 - E(D_{.:t}) + E(D_{.:t-1})] \geq 0.
\end{align*}
\]

The second equality follows from Assumption 6. This proves the result.

### B.6 Proof of Proposition 3

First, notice that under Assumption 8, \( S = \{D = 1\} \) and \( s_{g,1} = 1 \) for all \( g \). Therefore, for all \( (k, g) \in \{1, 2, 3\} \times \{0, \ldots, \bar{g}\} \),

\[
w_{k,g,1} = \omega_{k,g,1} = \frac{\epsilon_{k,g,1} E(D)}{E(D_{\epsilon_{k,G,T}})}.
\]

Moreover,

\[
P(\Omega_k < 0|S) = P(W_k < 0|D = 1, T = 1) = P(W_k < 0|D = 1).
\]

Let us reason by contradiction and suppose that \( P(W_k < 0|D = 1, T = 1) = 0 \) is equal to zero. Let us also suppose that \( E(D_{\epsilon_{k,G,T}}) > 0 \) (the proof is symmetric if instead \( E(D_{\epsilon_{k,G,T}}) < 0 \)). Then:

\[
0 = P(W_k < 0|D = 1, T = 1) \\
= P(\epsilon_{k,G,1} < 0|D = 1, T = 1) \\
= \frac{E[1\{\epsilon_{k,G,1} < 0\}|T = 1]}{E(D_{.:1})} \\
= \frac{E[E(D|G, T = 1)1\{\epsilon_{k,G,1} < 0\}|T = 1]}{E(D_{.:1})} \\
\geq \frac{\min_g E(D_{g,1})}{E(D_{.:1})} P(\epsilon_{k,G,1} < 0|T = 1).
\]

The second equality follows from the fact that if \( E(D_{\epsilon_{k,G,T}}) > 0 \), \( W_k \) is equivalent to \( 1\{\epsilon_{k,G,1} < 0\} \) conditional on \( T = 1 \). The fourth equality follows from the law of iterated expectations. The fifth equality follows from the fact that \( E(D|G, T = 1) \geq \min_g E(D_{g,1}) \).

The previous display implies that \( P(\epsilon_{k,G,1} < 0|T = 1) = 0 \). By definition of \( \epsilon_{k,g,t} \), \( E(\epsilon_{k,G,1}|T = 1) = 0 \). Hence, \( P(\epsilon_{k,G,1} = 0|T = 1) = 1 \). But then

\[
E(D_{\epsilon_{k,G,T}}) = E(D_{\epsilon_{k,G,1}}1\{T = 1\}) = 0.
\]
This implies that $\beta_1$ is not well defined, a contradiction.

B.7 Proof of Theorem 2

The reasoning is exactly the same as in Theorem 1, except that we rely on Lemma 1 instead of Corollary 3.

B.8 Proof of Theorem 3

**Proof of the first statement for $\beta_1$.**

First, remark that $\beta_1^X$ is the coefficient of $E(D|G,T)$ in the regression of $Y - X'\gamma_1$ on group and time fixed effects and $E(D|G,T)$. Therefore, by the Frisch-Waugh theorem,

$$
\beta_1^X = \frac{\text{cov}(\varepsilon_{1,G,T}, Y - X'\gamma_1)}{\text{cov}(\varepsilon_{1,G,T}, E(D|G,T))}.
$$

Then, reasoning as in the proof of Theorem 1, we obtain

$$
\text{cov}(\varepsilon_{1,G,T}, Y - X'\gamma_1) = E[\varepsilon_{1,G,T}DID_{Y - X'\gamma_1}(G,0,T,0)].
$$

Now, under Assumptions 1X$_1$ and 9, we can follow the same steps as those used to establish the first point of Corollary 3 to show that

$$
DID_{Y - X'\gamma_1}(g,0,t,0) = E(D_{g,t})\Delta_{g,t}^{TR} - E(D_{g,0})\Delta_{g,0}^{TR} - (E(D_{0,t})\Delta_{0,t}^{TR} - E(D_{0,0})\Delta_{0,0}^{TR}).
$$

Then, the proof follows exactly as that of the first statement of Theorem 1 for $\beta_1$.

**Proof of the first statement for $\beta_2$.**

First, remark that $\beta_2^X$ is the coefficient of $FD_D(G,T)$ in the regression of $FD_{Y - X'\gamma_2}(G,T)$ on time fixed effects and $FD_D(G,T)$. Therefore, by Frisch-Waugh theorem,

$$
\beta_2^X = \frac{\text{cov}(\varepsilon_{2,G,T}, FD_{Y - X'\gamma_2}(G,T)|T \geq 1)}{\text{cov}(\varepsilon_{2,G,T}, FD_D(G,T)|T \geq 1)}.
$$

Then, reasoning as in the proof of Theorem 1, we obtain

$$
\text{cov}(\varepsilon_{2,G,T}, FD_{Y - X'\gamma_2}(G,T)|T \geq 1) = E[\varepsilon_{2,G,T}DID_{Y - X'\gamma_2}(G,0,T,T-1)|T \geq 1].
$$

Now, as above with $k = 1$,

$$
DID_{Y - X'\gamma_2}(g,0,t,t-1) = E(D_{g,t})\Delta_{g,t}^{TR} - E(D_{g,t-1})\Delta_{g,t-1}^{TR} - (E(D_{0,t})\Delta_{0,t}^{TR} - E(D_{0,t-1})\Delta_{0,t-1}^{TR}).
$$

Then, the proof follows exactly as that of the first statement of Theorem 1 for $\beta_2$. 49
Proof of the first statement for $\beta_3$.

$\beta_3^X$ is equal $\lambda_3^X/\alpha_3^X$, where $\lambda_3^X$ (resp. $\alpha_3^X$) denotes the coefficient of $T \times f(G)$ in the regression of $Y$ (resp. $D$) on a constant, group fixed effects, $1\{T = 1\}$, $X$, and $T \times f(G)$. Then, remark that $\lambda_3^X$ (resp. $\alpha_3^X$) is the coefficient of $T \times f(G)$ in the regression of $Y - X'\gamma_3$ (resp. $D - X'\gamma_4$) on a constant, group fixed effects, $1\{T = 1\}$, and $T \times f(G)$. Therefore, it follows from the Frisch-Waugh theorem and the definition of $\varepsilon_{3,G,T}$ that

$$
\beta_3^X = \frac{\text{cov}(Y - X'\gamma_3, \varepsilon_{3,G,T})}{\text{cov}(D - X'\gamma_4, \varepsilon_{3,G,T})}.
$$

(31)

Then, adapting as above the reasoning in the proof of Theorem 1, we obtain

$$
\text{cov}(Y - X'\gamma_3, \varepsilon_{3,G,T}) = E[\varepsilon_{3,G,T}D ID_Y - X'\gamma_3(G, 0, 1, 0)] = E[\varepsilon_{3,G,T}D \Delta_{G,T}^{TR}].
$$

(32)

Now, we have, by Assumption 1X3,

$$
\text{cov}(D - X'\gamma_4, \varepsilon_{3,G,T}) = \text{cov}(D, \varepsilon_{3,G,T}) = E[\varepsilon_{3,G,T}D].
$$

(33)

Plugging Equations (32) and (33) into Equation (31) yields the result.

Proof of the second statement.

We only sketch the proof of the result for $\beta_1$. As above, the reasoning is similar to that in the proof of the second statement of Theorem 1, but replacing $Y$ by $Y - X'\gamma_1$. In particular, under Assumptions 1X1, 3, and 9, one can follow the same steps as those used to establish the second point of Corollary 3 to show that

$$
D ID_Y - X'\gamma_1(g, 0, t, t - 1) = (E(D_{g,t}) - E(D_{g,t-1})) \Delta_{g,t}^S - (E(D_{0,t}) - E(D_{0,t-1})) \Delta_{0,t}^S.
$$

B.9 Proof of Theorem 4

Proof of the first statement.

For all $(g^*, t) \in \{-1, 0, 1\} \times \{1, \ldots, \bar{t}\}$, let $G_{g^*,t} = \{g : \text{sgn}(FD_D(g, t)) = g^*\}$. First, note that for all $t \geq 1$,

$$
E(Y|G_t^* = 1, T = t) = \sum_{g \in G_{1,t}} r(g|1, t, t)E(Y_{g,t}).
$$

Similarly,

$$
E(QY|G_t^* = 1, T = t - 1) = \sum_{g \in G_{1,t}} r(g|1, t, t - 1)\frac{r(g|1, t, t)}{r(g|1, t, t - 1)}E(Y_{g,t-1})
= \sum_{g \in G_{1,t}} r(g|1, t, t)E(Y_{g,t-1}).
$$
Hence,
\[ \text{DID}_Y^*(1,t) = \sum_{g \in \mathcal{G}_{1,t}} r(g|1,t,t) [E(Y_{g,t}) - E(Y_{g',t-1}) - (E(Y|G_t^* = 0, T = t) - E(QY|G_t^* = 0, T = t - 1))]. \]

Moreover, reasoning as above,
\[ E(Y|G_t^* = 0, T = t) - E(QY|G_t^* = 0, T = t - 1) = \sum_{g' \in \mathcal{G}_{0,t}} r(g'|0,t,t) (E(Y_{g',t}) - E(Y_{g',t-1})). \]

Thus,
\[ \text{DID}_Y^*(1,t) = \sum_{(g,g') \in \mathcal{G}_{1,t} \times \mathcal{G}_{0,t}} r(g|1,t,t) r(g'|0,t,t) \text{DID}_Y(g, g', t, t - 1), \tag{34} \]

where \( \text{DID}(g, g', t, t - 1) \) is defined as above in Lemma 1. By definition of \( \mathcal{G}_{0,t} \), we have that under Assumption 2, for all \( g' \in \mathcal{G}_{0,t} \), \( P(S_{g',t}) = 0 \). Then, by Corollary 3, we have, for all \((g, g') \in \mathcal{G}_{1,t} \times \mathcal{G}_{0,t}\),
\[ \text{DID}_Y(g, g', t, t - 1) = P(S_{g,t}) \Delta_g^{S_{g,t}}. \]

Combining this equation with (34), we obtain
\[ \text{DID}_Y^*(1,t) = \sum_{g \in \mathcal{G}_{1,t}} r(g|1,t,t) P(S_{g,t}) \Delta_g^{S_{g,t}}. \]

Then, because \( P(G_t^* = 1, T = t) r(g|1,t,t) = P(G = g, T = t) \) for all \( g \in \mathcal{G}_{1,t} \),
\[ \frac{w_{1,t}}{\text{DID}_D^*(1,t)} \frac{\text{DID}_Y^*(1,t)}{\text{DID}_D^*(1,t)} = \frac{\sum_{g \in \mathcal{G}_{1,t}} P(G = g, T = t) P(S_{g,t}) \Delta_g^{S_{g,t}}}{\sum_{t'=1}^T \sum_{g' \in \{-1,1\}} g' \text{DID}_D^*(g', t') P(G_t^* = g', T = t')} \]

Reasoning as above, we obtain
\[ \text{DID}_Y^*(-1,t) = - \sum_{g \in \mathcal{G}_{-1,t}} r(g - 1, t, t) P(S_{g,t}) \Delta_g^{S_{g,t}}, \]
\[ \text{DID}_D^*(1,t) = \sum_{g \in \mathcal{G}_{1,t}} r(g|1,t,t) P(S_{g,t}), \]
\[ \text{DID}_D^*(-1,t) = - \sum_{g \in \mathcal{G}_{-1,t}} r(g - 1, t, t) P(S_{g,t}). \]

Combining all these and the fact that \( P(S_{g,t}) = 0 \) for all \( g \in \mathcal{G}_{0,t} \) yields
\[ \sum_{g^* \in \{-1,1\}} w_{g^*,t} \frac{\text{DID}_Y^*(g^*, t)}{\text{DID}_D^*(g^*, t)} = \frac{\sum_{g=0}^T P(G = g, T = t) P(S_{g,t}) \Delta_g^{S_{g,t}}}{\sum_{t'=1}^T \sum_{g' = 0}^T P(G = g', T = t) P(S_{g',t})}. \]

The result follows by summing over \( t \in \{1, \ldots, \bar{t}\} \) and by definition of \( \Delta^S \).
\textbf{Proof of the second statement.}

First, reasoning as above,

\begin{equation}
E(Y|G_t^* = 1, T = t) - E(QY|G_t^* = 1, T = t - 1) = \sum_{g \in \mathcal{G}_{1,t}} r(g|1, t, t)[E(Y_{g,t}) - E(Y_{g,t-1})] \tag{35}
\end{equation}

Moreover,

\begin{align*}
E(Y_{g,t}) - E(Y_{g,t-1}) \\
= & E(Y_{g,t}(1) - Y_{g,t}(0)|S)P(S_{g,t}) + E(Y_{g,t}(1)|D(t - 1) = 1)P(D(t - 1) = 1|G = g, T = t) \\
& - E(Y_{g,t-1}(1)|D(t - 1) = 1)P(D(t - 1) = 1|G = g, T = t - 1) \\
& + E(Y_{g,t}(0)|D(t - 1) = 0)P(D(t - 1) = 0|G = g, T = t) \\
& - E(Y_{g,t-1}(0)|D(t - 1) = 0)P(D(t - 1) = 0|G = g, T = t - 1) \\
= & \Delta_{g,t} P(S_{g,t}) + E(Y_{g,t}(1) - Y_{g,t-1}(1)|D(t - 1) = 1)E(D_{g,t-1}) \\
& + E(Y_{g,t}(0) - Y_{g,t-1}(0)|D(t - 1) = 0)(1 - E(D_{g,t-1})). \tag{36}
\end{align*}

where the first equality follows from Assumption 2.3 and the second from Assumption 2.1 and 2.2. Now, by a similar reasoning as that used to obtain (35),

\begin{align*}
E(Q\delta_D|G_t^* = 1, T = t - 1) &= E(Q(D\delta_1 + (1 - D)\delta_0|G_t^* = 1, T = t - 1) \\
&= \sum_{g \in \mathcal{G}_{1,t}} r(g|1, t, t)[E(D_{g,t-1})\delta_1 + (1 - E(D_{g,t-1}))\delta_0]. \tag{37}
\end{align*}

Next, we have, for any \(g \in \mathcal{G}_{1,t},\)

\begin{align*}
\delta_d &= E(Y|D = d, G_t^* = 0, T = t) - E(Q_d Y|D = d, G_t^* = 0, T = t - 1) \\
&= \sum_{g' \in \mathcal{G}_{0,t}} r_d(g'|t, t)E(Y_{g',t}(d) - Y_{g',t-1}(d)|D(t - 1) = d) \\
&= \sum_{g' \in \mathcal{G}_{0,t}} r_d(g'|t, t)E(Y_{g,t}(d) - Y_{g,t-1}(d)|D(t - 1) = d) \\
&= E(Y_{g,t}(d) - Y_{g,t-1}(d)|D(t - 1) = d).
\end{align*}

The first equality follows from a similar reasoning as that used to obtain (35). The third follows from Assumption 1'.

Therefore, in view of (35), (36) and (37), we obtain

\begin{align*}
E(Y|G_t^* = 1, T = t) - E(QY|G_t^* = 1, T = t - 1) \\
= \sum_{g \in \mathcal{G}_{1,t}} r(g|1, t, t)\Delta_{g,t}^S P(S_{g,t}) + E(Q\delta_D|G_t^* = 1, T = t - 1).
\end{align*}
Rearranging the previous display and using the definition of $TCD^*(1, t)$ yields

$$TCD^*(1, t) = \sum_{g \in G_{1, t}} r(g|1, t, t) P(S_{g,t}) \Delta_{g,t}^S.$$ 

A similar reasoning yields

$$TCD^*(-1, t) = -\sum_{g \in G_{-1, t}} r(g|-1, t, t) P(S_{g,t}) \Delta_{g,t}^S.$$ 

The rest of the proof is the same as that of the first statement.