Inequality, Business Cycles, and Monetary-Fiscal Policy*

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June 6, 2018

Abstract

We study optimal monetary and fiscal policy in a model with heterogeneous agents, incomplete markets, and nominal rigidities. We develop numerical techniques to approximate Ramsey plans and apply them to a calibrated economy to compute optimal responses of nominal interest rates and labor tax rates to aggregate shocks. Responses differ qualitatively from those in a representative agent economy and are an order of magnitude larger. Taylor rules poorly approximate the Ramsey optimal nominal interest rate. Conventional price stabilization motives are swamped by an across person insurance motive that arises from heterogeneity and incomplete markets.

Key words: Sticky prices, heterogeneity, business cycles, monetary policy, fiscal policy

*We thank Adrien Auclert, Benjamin Moll, and seminar participants at numerous seminars and conferences for helpful comments

1 Introduction

We study monetary and fiscal policy in a New Keynesian economy populated by agents who face aggregate and idiosyncratic risks. Agents differ in wages, exposures to aggregate shocks, holdings of financial assets, and abilities to trade assets. Incomplete financial markets prevent agents from fully insuring risks. Firms are monopolistically competitive. Price adjustments are costly. We examine how the Ramsey planner's choices of nominal interest rates, transfers, and proportional labor tax rates respond to aggregate shocks.

Analysis of Ramsey policies in settings like ours faces substantial computational challenges. The aggregate state in a recursive formulation of the Ramsey problem includes the joint distribution of individual asset holdings and auxiliary promise-keeping variables that had been chosen by the planner earlier. The law of motion for that high-dimensional object must be jointly determined with the optimal policies. This feature renders inapplicable existing numerical techniques that rely on approximating around a known invariant distribution associated with fixed government policies.

To overcome this challenge, we develop a new computational approach that can be applied to economies with substantial heterogeneity and does not require knowing their long-run properties in advance. Our approach builds on a perturbation theory that uses small-noise expansions with respect to a one-dimensional parameterization of uncertainty. Each period along a sample path, we apply a perturbation algorithm evaluated at the current cross-sectional distribution to approximate policy functions. We use approximate decision rules for the current period to determine outcomes including the cross-sectional distribution next period. Then we obtain approximations of next period's decision rules by perturbing around that new distribution. In this way we update points around which policy functions are approximated along the equilibrium path.

Our perturbation approach requires computing derivatives of policy functions with respect to all state variables. One state variable is a distribution over a multi-dimensional space of agents' characteristics. Except for very simple models of heterogeneity, it is impractical to compute the derivative with respect to this distribution (i.e., the Frechet derivative). We make progress by showing that our expansion requires only the value of that derivative at a point that corresponds to the optimal Ramsey response. We derive an expression for this value and show that it is a linear function of variables that are easy to compute. That mathematical structure allows us to compute approximations to each agent's policy function by solving low dimensional linear systems of equations that are independent across agents. This feature enables us to solve our model quickly even though it has ample sources of heterogeneity. Furthermore, we proceed to show that the same computationally convenient linear structure is preserved for second- and higher-order expansions. That allows us to capture precautionary and hedging motives.

We apply our approach to a textbook New Keynesian sticky price model (see, e.g., Galí (2015)) augmented with heterogeneous agents in the spirit of Bewley-Aiyagari. Agents' wages are subject to idiosyncratic and aggregate shocks that we calibrate to match empirical facts about labor earnings documented by Storesletten et al. (2004) and Guvenen et al. (2014). We make the initial joint distribution of assets and wages match cross-sectional moments in the Survey of Consumer Finances. We posit two types of aggregate shocks: a productivity shock and a shock to the elasticity of substitution between differentiated intermediate goods that affects firms' optimal markups. Financial markets are incomplete and agents can trade only non-state-contingent nominal debt. We study two types of Ramsey policies. First, a "purely monetary policy" planner is required to keep the labor tax rate fixed, an assumption commonly used with New Keynesian models. In this case, the planner can adjust only nominal interest rates and a uniform lump-sum transfer. Second, a more powerful "monetary-fiscal" Ramsey planner can adjust tax rates in addition to interest rates and transfers.

Two motives inherited from the Ramsey planner's objective function shape optimal policies. One is the usual New Keynesian motive to use government policies to offset inefficiencies caused by sticky prices. A second is a desire to provide insurance. Without a complete set of Arrow securities, agents cannot hedge risk, so aggregate shocks affect different agents differentially. A planner can use monetary and fiscal tools to even out the impact of those shocks and provide insurance. In our calibrated economy, we find that the insurance motive quantitatively swamps price stability considerations. That makes optimal policy responses differ significantly from those in a representative agent economy where they are primarily driven by the price stability motive.

Consider first the optimal monetary response to a positive markup shock. One effect of a markup shock is that firms want to increase their prices. Nominal rigidities make that costly. When price stability is its only motive, the planner increases the nominal interest rate in response to a positive markup shock. That lowers aggregate demand and marginal costs and thereby reduces firms' incentives to raise prices. Galí (2015) dubs this optimal response in representative agent New Keynesian economies as "leaning against the wind". With heterogeneous agents, a markup shock redistributes resources among households. Higher markups shift factor income from wages to dividends, benefiting firm owners and hurting wage recipients. To provide insurance, the planner can decrease interest rates in order to boost aggregate demand and real wages. The planner does that to offset the negative effect of the markup shock on wage-earners. The net effect of the markup shock on optimal policy depends on the relative strengths of the planner's price stability and insurance motives. When we calibrate the distribution of equity ownership to U.S. data, we find that a one standard deviation positive markup shock, which implies a 0.5 percentage point increase in firms' optimal markups, calls for a 0.3 percentage point decrease in the nominal interest rate versus a 0.05 percentage point increase with a representative agent calibration.

Price stability and insurance motives also bring different prescriptions for the optimal fiscal response. To ensure price stability in response to a positive markup shock, the planner wants to lower the marginal tax rate one for one with the shock. This requires lowering transfers to satisfy the government's budget constraint, making the average taxes more regressive. To provide insurance, the planner needs to raise taxes and transfers to increase progressivity and offset the distributionary impact of the shock. In the calibrated economy, the insurance channel dominates: in response to a one standard deviation positive markup shock, a planner increases tax rates by 2 percentage points. This fiscal response comes on top of a cut in nominal interest rates by about 0.25 percentage points and a 1 percentage point spike in inflation. The tax increase is short-lived, which contrasts with the very persistent effects of aggregate shocks on optimal tax rates in non-monetary economies, such as Barro (1979) and Aiyagari et al. (2002). The reason for this is that deadweight losses from temporary tax changes in New Keynesian models are lower than those from permanent tax changes: firms choose not to adjust nominal prices to temporary shocks, and consequently a transitory tax change mostly affects tax incidence but not the labor wedge.

The insurance motive also substantially affects Ramsey responses to TFP shocks.

With heterogeneous agents, effects of productivity shocks are not shared equally: borrowers in state-non-contingent nominal debt are hurt more than lenders. This effect is further amplified when, as in the data, financial and wage income are positively correlated and because labor incomes of low earners are especially adversely affected by recessions (see Guvenen et al. (2014)). To even out adverse effects of the shock, the planner cuts nominal interest rates, thereby generating inflation and lowering realized returns on debt holdings. This contrasts with a typical New Keynesian response that would maintain stable prices by adjusting nominal rates one-for-one with the "natural" rate of interest.[1] In our calibrated economy, we find that in response to a productivity shock that lowers output by 3 percentage point, nominal rates are lowered by 0.6 percentage point.

In our heterogeneous agent economy, Taylor rules perform substantially worse than in a representative agent counterpart. Without heterogeneity, the main purpose of optimal policy is to stabilize inflation; Taylor rules meet this objective well. But they do poorly in the presence of heterogeneity for two reasons. First, in response to markup shocks it is optimal to move interest rates in the opposite direction from that prescribed by standard Taylor rules. Second, while Taylor rules make nominal interest rates and inflation as persistent as exogenous shocks, the Ramsey planner prefers more transient nominal interest rates and inflation to provide insurance.

We also investigate the effects of trading frictions on optimal policy. Typical Bewley-Aiyagari models poorly match observed cross-section distributions of MPCs. We alter our economy to include a set of agents who can hold assets and consume their returns but cannot trade them. We choose the distribution of such agents to match MPC heterogeneity estimated by Jappelli and Pistaferri (2014). We find that in such an economy the optimal response of nominal interest rates is smaller than in our benchmark calibration to markup shocks, but not to productivity shocks. This outcome emerges because in response to a markup shock the Ramsey planner would like to alter aggregate demand and real wages. These quantities are determined by *average* actions across agents. Changes in interest rates have a direct effect only on agents who can trade their assets. By reducing the number of such agents, trading frictions make monetary policy less effective, an outcome emphasized by Auclert (2017) and Kaplan et al. (2018). But in response to a TFP shock the planner wants

[1]The natural interest rate is defined as the real interest rates that would prevail in an economy without nominal rigidities.

to affect returns on financial assets that are determined by a *marginal* investor and do not depend on the number of such investors. We also find that the presence of trading frictions increases the importance of fiscal instruments like transfers that directly affect the consumption of high MPC agents.

1.1 Earlier works

A number of contributions characterized and approximated competitive equilibria with exogenously fixed government policies. These employ one of two broad approaches that either (i) compute non-linear policies after summarizing the distributional state variable with a low dimensional vector of its moments, for instance Krusell and Smith (1998), or (ii) apply a first-order Taylor expansion of policy functions with respect to the aggregate shocks around an invariant distribution for an economy without aggregate shocks, for instance Reiter (2009).[2] Neither approach is suitable for our problem. First, for us an invariant distribution depends on a key object that we want to compute, namely, the Ramsey policy. Furthermore, the invariant distribution under a Ramsey plan for an incomplete markets without aggregate shocks looks very different from the invariant distribution of the same economy with small aggregate shocks. In addition, the speed of convergence to this invariant distribution is slow and the behavior of optimal policies around the invariant distribution differs significantly from the behavior of policies away from the invariant distribution.[3] Finally, the state space in an economy under the Ramsey plan is much larger because it includes promise-keeping variables for the planner in addition to all of states that appear in a corresponding economy with an arbitrary exogenous government policy.

Our computational approach uses a one dimensional scaling of uncertainty and a sequence of small-noise expansions along an equilibrium path. Our approach is most closely related to Fleming (1971), Fleming and Souganidis (1986), and Anderson et al. (2012). State variables in the models studied by Anderson et al. are low dimensional, for example just an aggregate capital stock. In contrast, our state variable is a joint

[2] See Algan et al. (2014) for variants of the Krusell and Smith method and Ahn et al. (2017); Winberry (2016); Childers et al. (2018) for the variants of the Reiter method.

[3] Such outcomes are evident in Barro (1979), Aiyagari et al. (2002), and Farhi (2010). In Bhandari et al. (2017a), we consider a simplified version of the economy studied in this paper and show that while the invariant distribution in a Ramsey problem without aggregate shocks is pinned down by initial conditions (such as the initial distribution of wealth), the invariant distribution in a corresponding problem with aggregate shocks is determined by motives to hedge such shocks. Therefore, the invariant distribution is discontinuous in the degree of aggregate risk around zero risk.

distribution over a multi-dimensional domain. That renders direct application of the Anderson et al. technique computationally infeasible. We overcome this problem by proving and applying a factorization theorem (see section 3) that allows us to break one big problem into a manageable collection of much smaller ones. We show how these ideas can be used to construct second- and higher-order approximations. Our approach builds on Evans (2015).

Most studies of optimal policy have been restricted to economies with very limited or no heterogeneity. Schmitt-Grohe and Uribe (2004) and Siu (2004) are two notable quantitative studies of Ramsey policies in representative agent New Keynesian models. Our parameterization follows theirs. So as might be expected, when all heterogeneity is shut down, we obtain quantitative results similar to theirs. Bilbiie and Ragot (2017), Nuno and Thomas (2016), Challe (2017), and Debortoli and Gali (2017) study optimal monetary policy in economies with very limited heterogeneity. In those settings, the aggregate distribution disappears from the formulation of the Ramsey problem and the analysis can be done using traditional techniques. Like us, they emphasize that uninsurable risk creates reasons for a planner to depart from price stability. A recent paper by Legrand and Ragot (2017) develops a method different from ours to approximate Ramsey allocations in incomplete market economies with heterogeneity. They apply their method to a neoclassical economy.

2 Environment

Our economy is populated by a continuum of infinitely lived households. Individual i's preferences over final consumption good $\{c_{i,t}\}_t$ and hours $\{n_{i,t}\}_t$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c^{1-\nu}}{1-\nu} - \frac{n^{1+\gamma}}{1+\gamma} \right), \tag{1}$$

where $\mathbb{E}_t$ is an expectations operator conditioned on time t information and $\beta \in (0,1)$ is a time discount factor.

Agent i who works $n_{i,t}$ hours supplies $\epsilon_{i,t} n_{i,t}$ units of effective labor, where $\epsilon_{i,t}$ is an exogenous productivity process. Effective labor receives nominal wage $P_t W_t$, where P_t is the nominal price of the final consumption good at time t. Labor income is taxed at a proportional labor tax rate Υ_t. All agents receive a uniform lump sum transfer T_tP_t. Agents trade a one-period risk-free nominal bond with price Q_t in units of the final consumption good. We use $P_tb_{i,t}$, P_tB_t to denote bond holdings of agent i and the debt position of the government respectively. Π_t denotes the net inflation rate. Finally, $d_{i,t}$ denotes dividends from intermediate goods producers measured in units of the final good. Agent i's budget constraint at t is

$$c_{i,t} + Q_t b_{i,t} = (1 - \Upsilon_t) W_t \epsilon_{i,t} n_{i,t} + T_t + d_{i,t} + \frac{b_{i,t-1}}{1 + \Pi_t}. \tag{2}$$

The government's budget constraint at time t is

$$\bar{G} + T_t + \frac{1}{1 + \Pi_t} B_{t-1} = \Upsilon_t W_t \int_i \epsilon_{i,t} n_{i,t} di + Q_t B_t,$$

where $\bar{G}$ is the level of non-transfer expenditures.

A final good Y_t is produced by competitive firms that use a continuum of intermediate goods $\{y_t(j)\}_{j\in[0,1]}$ in a production function

$$Y_t = \left[\int_0^1 y_t(j)^{\frac{\Phi_t - 1}{\Phi_t}} dj \right]^{\frac{\Phi_t}{\Phi_t - 1}},$$

where the elasticity of substitution Φ_t is stochastic. Final good producers take final good prices P_t and intermediate goods prices $\{p_t(j)\}_j$ as given and solve

$$\max_{\{y_t(j)\}_{j\in[0,1]}} P_t \left[\int_0^1 y_t(j)^{\frac{\Phi_t - 1}{\Phi_t}} dj \right]^{\frac{\Phi_t}{\Phi_t - 1}} - \int_0^1 p_t(j) y_t(j) dj. \tag{3}$$

Outcomes of optimization problem (3) are a demand function for intermediate goods

$$y_t(j) = \left(\frac{p_t(j)}{P_t} \right)^{-\Phi_t} Y_t,$$

and a nominal price satisfying

$$P_t = \left(\int_0^1 p_t(j)^{1-\Phi_t} \right)^{\frac{1}{1-\Phi_t}}.$$

Intermediate goods $y_t(j)$ are produced by monopolists with production function

$$y_t(j) = \left[n_t^D(j)\right]^\alpha,$$

where $n_t^D(j)$ is effective labor hired by firm j and $\alpha \in (0,1]$. These intermediate goods monopolists face downward sloping demand curves $\left(\frac{p_t(j)}{P_t}\right)^{-\Phi_t} Y_t$ and choose prices $p_t(j)$ while bearing quadratic Rotemberg (1982) price adjustment costs $\frac{\psi}{2}\left(\frac{p_t(j)}{p_{t-1}(j)} - 1\right)^2$ measured in units of the final consumption good. Firm j chooses prices $\{p_t(j)\}_t$ that solve

$$\max_{\{p_t(j)\}_t} \mathbb{E}_0 \sum_t \beta^t \left(\frac{C_t}{C_0}\right)^{-\nu} \left\{ \left(\frac{p_t(j)}{P_t}\right)^{-\Phi_t} Y_t \left[\frac{p_t(j)}{P_t} - \frac{W_t}{\alpha}\left(\left(\frac{p_t(j)}{P_t}\right)^{-\Phi_t} Y_t\right)^{\frac{1-\alpha}{\alpha}}\right] - \frac{\psi}{2}\left(\frac{p_t(j)}{p_{t-1}(j)} - 1\right)^2 \right\}, \tag{4}$$

where we have imposed that each firm values profit streams with a stochastic discount factor that is driven by aggregate consumption $C_t = \int c_{i,t} di$.[4]

In a symmetric equilibrium, $p_t(j) = P_t$, $y_t(j) = Y_t$ for all j. Market clearing conditions in labor, goods, and bond markets are:

$$N_t = \int \epsilon_{i,t} n_{i,t} di, \qquad D_t = Y_t - W_t N_t - \frac{\psi}{2}\Pi_t^2, \tag{5}$$

$$y_t(j) = Y_t = N_t^\alpha, \tag{6}$$

$$C_t + \bar{G} = Y_t - \frac{\psi}{2}\Pi_t^2, \tag{7}$$

$$\int_i b_{i,t} di = B_t. \tag{8}$$

There are aggregate and idiosyncratic shocks. Aggregate shocks are a "markup" shock[5] Φ_t and an aggregate productivity Θ_t. The markup shock follows an AR(1)

[4] In economies with heterogeneous agents and incomplete markets one has to take a stand on how firms are valued. Using stochastic discount factor induced by aggregate consumption makes our exposition most transparent. We have experimented with other alternatives such as weighted means of individual intertemporal marginal rates of substitutions and found them to have negligible effects on the main results.

[5] The New Keynesian literature has utilized markup or cost-push shocks to account for patterns observed during business cycles (see, e.g., Smets and Wouters (2007)). The literature has proceeded to study implications of those shocks for optimal monetary policy in a representative agent framework

stochastic process

$$\ln \Phi_t = \rho_\Phi \ln \Phi_{t-1} + (1-\rho_\Phi) \ln \bar{\Phi} + \mathcal{E}_{\Phi,t},$$

where $\mathcal{E}_{\Phi,t}$ is mean-zero and i.i.d. over time. We consider two alternative specification for productivity shocks. In the first specification, we simply assume that $\ln \Theta_t$ follows an AR(1) process described by

$$\ln \Theta_t = \rho_\Theta \ln \Theta_{t-1} + (1-\rho_\Theta) \ln \bar{\Theta} + \mathcal{E}_{\Theta,t}, \tag{9a}$$

where $\mathcal{E}_{\Theta,t}$ is mean zero and i.i.d. over time. In the second specification we assume that $\ln \Theta_t$ follows a growth rate shock process

$$\ln \Theta_t = \ln \Theta_{t-1} + \mathcal{E}_{\Theta,t}, \tag{9b}$$

where again $\mathcal{E}_{\Theta,t}$ is mean zero and i.i.d. over time. Under the growth rate specification, we scale government expenditures at time t to equal $G_t = \bar{G}\Theta_t$ where $\bar{G}$ is a non-negative constant and menu cost so that $\psi_t = \bar{\psi}\Theta_t$ for some non-negative constant $\bar{\psi}$ to achieve stationarity.[6]

Individual productivity $\epsilon_{i,t}$ follows a stochastic process described by

$$\ln \epsilon_{i,t} = \ln \Theta_t + \ln \theta_{i,t} + \varepsilon_{\epsilon,i,t}, \tag{10}$$

$$\ln \theta_{i,t} = \rho_\theta \ln \theta_{i,t-1} + f(\theta_{i,t-1}) \mathcal{E}_{\Theta,t} + \varepsilon_{\theta,i,t}, \tag{11}$$

where $\varepsilon_{\epsilon,i,t}$ and $\varepsilon_{\theta,i,t}$ are mean-zero, uncorrelated with each other and i.i.d. over time. This specification of idiosyncratic shocks builds on formulations of wage dynamics used by Storesletten et al. (2001) and Low et al. (2010)) in which $\varepsilon_{\epsilon,i,t}$ and $\varepsilon_{\theta,i,t}$ correspond to transitory and persistent shocks to individual productivities. We augment this specification with a function $f(\theta_{i,t-1})$ that makes an aggregate productivity shock have different loadings for agents with different earning histories. Doing that allows

(see, e.g., Clarida et al. (2001), Galí (2015), Woodford (2003)). As in Galí (2015), we interpret these shocks as changes in desired markups arising from fluctuations in the elasticity of substitution parameter Φ. We leave for future work the study of other shocks such as wage markups.

[6] General insights are similar from these two alternative specifications, but the i.i.d growth specification provides a convenient benchmark to study monetary policy because without market incompleteness and nominal frictions, the real interest rate is constant.

us to capture some of the facts documented by Guvenen et al. (2014).

We normalize the initial price level $P_{-1} = 1$. Agent i in period 0 is characterized by a triple $(\theta_{i,-1}, b_{i,-1}, s_i)$, where θ_i is agent i's persistent productivity component, $b_{i,-1}$ the initial holding of debt, and s_i the ownership of equity. Agent i's dividends in period t are given by $d_{i,t} = s_i D_t$. This imposes that agents do not trade equity and that s_i is a permanent characteristic. We refer to the collection $\{\theta_{i,-1}, b_{i,-1}, s_i\}_i$ as an initial condition.

Definition 1. Given an initial condition and a monetary-fiscal policy $\{Q_t, \Upsilon_t, T_t\}_t$, a competitive equilibrium is a sequence $\left\{\{c_{i,t}, n_{i,t}, b_{i,t}\}_i, C_t, N_t, B_t, W_t, P_t, Y_t, D_t\right\}_t$ such that: (i) $\{c_{i,t}, n_{i,t}, b_{i,t}\}_{i,t}$ maximize (1) subject to (2) and natural debt limits; (ii) final goods firms choose $\{y_t(j)\}_j$ to maximize (3); (iii) intermediate goods producers' prices solve (4) and satisfy $p_t(j) = P_t$; and (iv) market clearing conditions (5)-(8) are satisfied.

A utilitarian Ramsey planner orders allocations by

$$\mathbb{E}_0 \int \sum_{t=0}^{\infty} \beta^t \left[\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right] di. \tag{12}$$

Definition 2. Given an initial condition and a constant tax sequence $\{\Upsilon_t\}_t$ satisfying $\Upsilon_t = \bar{\Upsilon}$ for some $\bar{\Upsilon}$, *an optimal monetary policy* is a sequence $\{Q_t, T_t\}_t$ that supports a competitive equilibrium allocation that maximizes (12). Given an initial condition, *an optimal monetary-fiscal policy* is a sequence $\{Q_t, \Upsilon_t, T_t\}_t$ that supports a competitive equilibrium allocation that maximizes (12). A maximizing monetary or monetary-fiscal policy is called the *Ramsey plan*; an associated allocation is called the *Ramsey allocation.*

A few remarks about our baseline formulation. We begin by assuming natural debt limits. This provides us with a useful benchmark and also avoids issues about normative implications in economies with ad hoc debt limits being sensitive to assumptions one makes on a government's ability to enforce debt and tax repayments.[7]

[7] In Bhandari et al. (2017b) we provide a comprehensive treatment of a Ramsey problem with ad hoc debt limits. In the economy with ad hoc debt limits the planner can simply choose the timing of transfers to undo such ad hoc debt limits. If the planner enforces debt and tax liabilities equally then welfare in the economies with ad hoc and natural debt limits coincides. Welfare can sometimes be improved in the economy with ad hoc debt limits if the planner commits not to enforce private debt contracts (see also Yared (2013) for a related result).

After presenting results with natural debt limits, we introduce credit market frictions in section 6. We discipline these frictions by matching observed heterogeneity in the marginal propensities to consume and study how these frictions affect the Ramsey plans. Assuming that equity ownership is fixed provides us with a simple way to calibrate portfolio shares.[8] We have distinguished between purely monetary and mixed monetary-fiscal policies because doing so allows us to respect a common argument that institutional constraints make it difficult to adjust tax rates in response to typical business cycle shocks, leaving nominal interest rates as the government's only tool for ameliorating such shocks. We capture that argument by studying optimal monetary policy when tax rates $\{\Upsilon_t\}_t$ are fixed at some level $\bar{\Upsilon}$. The monetary-fiscal Ramsey plan evaluates the optimal policies when this restriction is dropped.

3 Solution method

Following steps used by Lucas and Stokey (1983), Schmitt-Grohe and Uribe (2004), and others it is straightforward to establish that a monetary-fiscal policy $\{Q_t, \Upsilon_t, T_t\}_t$ and $\left\{\{c_{i,t}, n_{i,t}, b_{i,t}\}_i, C_t, N_t, W_t, P_t, Y_t, D_t\right\}_t$ are a competitive equilibrium if and only if they satisfy (2), (5)-(7) and

$$(1-\Upsilon_t)W_t\epsilon_{i,t}c_{i,t}^{-\nu} = n_{i,t}^{\gamma}, \tag{13}$$

$$Q_{t-1}c_{i,t-1}^{-\nu} = \mathbb{E}_{t-1}c_{i,t}^{-\nu}\left(1+\Pi_t\right)^{-1}, \tag{14}$$

$$\frac{1}{\psi}Y_t\left[1-\Phi_t\left(1-\frac{W_t}{\alpha N_t^{\alpha-1}}\right)\right] - \Pi_t(1+\Pi_t) + \beta\mathbb{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\Pi_{t+1}(1+\Pi_{t+1}) = 0. \tag{15}$$

To formulate the Ramsey problem, it is convenient to rewrite some of these constraints. After defining $a_{i,t} \equiv b_{i,t}Q_tc_{i,t}^{-\nu}$, substituting for Q_t and $(1-\Upsilon_t)W_t$, and solving the budget constraint forward, equations (2) can be expressed as a sequence of measurability constraints inherited from the risk-free nature of one-period debt:

$$\frac{a_{i,t-1}c_{i,t}^{-\nu}(1+\Pi_t)^{-1}}{\beta\mathbb{E}_{t-1}c_{i,t}^{-\nu}(1+\Pi_t)^{-1}} = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(c_{i,t+s}^{1-\nu} - c_{i,t+s}^{-\nu}(T_{t+s}+s_iD_{t+s}) - n_{i,t+s}^{1+\gamma}\right)\right] \tag{16}$$

[8] More generally, one needs some adjustment costs to reconcile agents' portfolio choices with those observed in data. Exploring normative implications of alternative cost specifications is interesting but outside the scope of the paper.

for $t \geq 1$ as well as a similar constraint at $t = 0$ in which the left hand side is replaced with $b_{i,-1}c_{i,0}^{-\nu}$. Define

$$m_{i,t}^{1/\nu} \equiv \frac{c_{i,t}}{C_t} \tag{17}$$

and rewrite Euler equation (14) as

$$C_{t-1}^{-\nu} Q_{t-1} = \beta m_{i,t-1} \mathbb{E}_{t-1} c_{i,t}^{-\nu} (1 + \Pi_t)^{-1}. \tag{18}$$

Variable $m_{i,t}$ can be interpreted as a time-t Pareto-Negishi weight of agent i. It will serve as a state variable in a recursive formulation of the Ramsey problem.

We formulate the Ramsey problem using ideas of Marcet and Marimon (2011). Let $\beta^t \Lambda_t$, $\beta^t \xi_{i,t}$ be Lagrange multipliers on (15) and (16), respectively. Define $\mu_{i,0} \equiv \xi_{i,0}$, $\mu_{i,t} = \mu_{i,t-1} + \xi_{i,t}$. Given an initial condition $\{\theta_{i,-1}, b_{i,-1}, s_i\}_i$, and $\Lambda_{-1} = a_{i,-1} = 0$, the Ramsey monetary-fiscal problem solves:

$$\begin{aligned}\inf\sup \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \int \Bigg[& \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right) + \left(c_{i,t}^{1-\nu} - c_{i,t}^{-\nu}(T_t + s_i D_t) - n_{i,t}^{1+\gamma} \right) \mu_{i,t} \\ & - \frac{a_{i,t-1} c_{i,t}^{-\nu} (1+\Pi_t)^{-1} \xi_{i,t}}{\beta \mathbb{E}_{t-1} \left[c_{i,t}^{-\nu} (1+\Pi_t)^{-1} \right]} + (1-\beta) c_{i,0}^{-\nu} b_{i,-1} \mu_{i,0} di \\ & + \Lambda_t C_t^{-\nu} Y_t \left[1 - \Phi_t \left(1 - \frac{W_t}{\alpha N_t^{\alpha-1}} \right) \right] + (\Lambda_{t-1} - \Lambda_t) C_t^{-\nu} \psi \Pi_t (1+\Pi_t) \Bigg\}, \end{aligned} \tag{19}$$

subject to (5), (6), (7), (13), (17), (18) and $\mu_{i,t} = \mu_{i,t-1} + \xi_{i,t}$, where the sup is with respect to $\left\{ \{c_{i,t}, n_{i,t}, b_{i,t}\}_i, C_t, N_t, B_t, W_t, P_t, Y_t, D_t \right\}_t$, and inf is with respect to the multipliers $\{\mu_{i,t}, \Lambda_t\}_t$. The Ramsey monetary problem is the same except the sequence $\{\Upsilon_t\}_t$ is exogenously fixed.

We call the subsequence of the Ramsey plan for $t \geq 1$ a continuation Ramsey plan. It is recursive in aggregate states $(\Theta_{t-1}, \Phi_{t-1})$, the Lagrange multiplier on the Phillips curve equation Λ_{t-1}, and the joint *distribution* over $\boldsymbol{z}_{i,t-1} \equiv (m_{i,t-1}, \mu_{i,t-1}, \theta_{i,t-1}, s_i)$ that we denote by Ω_{t-1}. Time t policy functions for aggregate variables depend on $(\Theta_{t-1}, \Phi_{t-1}, \Omega_{t-1}, \Lambda_{t-1}, \mathcal{E}_{\Theta,t}, \mathcal{E}_{\Phi,t})$ while time t policy functions for individual variables also depend on $(z_{i,t-1}, \varepsilon_{\epsilon,i,t}, \varepsilon_{\theta,i,t})$. The Ramsey plan includes choices at time 0 and solves time $t = 0$ first-order conditions subject to the restriction that future variables conform to the continuation Ramsey plan.

Dependence of policy functions on the history-dependent endogenous distribution Ω makes computing the Ramsey plan challenging. The law of motion of the high-dimensional state variable Ω depends on optimal monetary-fiscal policies. As we discussed in the introduction, a key difficulty is that before optimal policies are computed little can be said about properties of the stochastic process Ω_t and its invariant distribution. This structure renders inapplicable computational techniques for solving heterogeneous agent economies that approximate around a stationary distribution associated a fixed set of government policies.

To overcome this difficulty we develop a new computational method that builds on the perturbation theory of Fleming (1971), Fleming and Souganidis (1986), and Anderson et al. (2012) as well as on Evans (2015). The method constructs a sequence of small-noise expansions of policy functions around pertinent time t state variables. Using this approach, we first approximate optimal policy functions at time t around the time-t state including Ω_{t-1}, then use those to compute Ω_t, and finally construct approximations of optimal policy functions at time $t+1$ around the time-$t+1$ state including Ω_t. To begin, we illustrate our approach in a simple setting. We next extend things to the more general setting of section 2. Detailed derivations, proofs, and other details are relegated to the online appendix where we also show how our method can be used to solve for a competitive equilibrium with fixed government policies and we use that economy to test the accuracy of our approximations by comparing our solution to one obtained with standard numerical methods.

3.1 Approximations in the simple economy

We focus here on a special case designed to show the main steps and insights transparently. We assume that all shocks are i.i.d. and that there is a single non-trivial aggregate shock $\mathcal{E}$ and a single non-trivial idiosyncratic shock ε (say, $\mathcal{E}_{\Theta,t}$ and $\varepsilon_{\theta,i,t}$). Furthermore, we assume that agents have identical equity holdings $s_i = 1$, in which case one can show that $\Lambda_t = 0$ for all t. These assumptions make Ω the only aggregate state variable for the $t \geq 1$ continuation Ramsey plan; Ω becomes a distribution over the two-dimensional space of $\boldsymbol{z} = (m, \mu)$. This is the minimal structure needed to illustrate our method.[9]

[9]The distribution of states Ω is defined over a two-dimensional space because the Ramsey planner needs to keep track of two variables for each agent: a variable capturing agent's current assets and a variable capturing past implicit promises of the planner which comes from the need to search for optimal policies. In the recursive formulation of competitive equilibrium with fixed policies, Ω is a distribution over a one dimensional space of agents' Pareto-Negishi weights (or, equivalently, their

We use tildes to denote policy functions in the $t \geq 1$ continuation Ramsey plan. The aggregate policy functions consist of all upper-case choice variables in problem (19) as well as Lagrange multipliers on aggregate resource constraints (5)-(7). We denote the vector of these functions by $\tilde{\boldsymbol{X}}$. Individual policy functions consist of all lower-case choice variables in problem (19) as well as Lagrange multipliers on individual constraints (13), (17), (18); we denote the vector of these variables by $\tilde{\boldsymbol{x}}$. Policy functions for individual states $\tilde{\boldsymbol{z}}$ are included in the vector $\tilde{\boldsymbol{x}}$ and we define p to be the selection matrix that returns $\tilde{\boldsymbol{z}}$ from $\tilde{\boldsymbol{x}}$, i.e., $\tilde{\boldsymbol{z}} = \mathsf{p}\tilde{\boldsymbol{x}}$.

Consider the full set of first-order optimality conditions to problem (19). These conditions can be split into two groups. The first group consists of the optimality conditions for individual policy functions. They show the relationship between current period individual and aggregate policy functions $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{X}}$, current period realizations of shocks ε, $\mathcal{E}$, and expectations of current and next period policy functions $\mathbb{E}\left[\tilde{\boldsymbol{x}}|\boldsymbol{z},\Omega\right]$, $\mathbb{E}\left[\tilde{\boldsymbol{x}}(\cdot,\cdot,\tilde{\boldsymbol{z}}(\boldsymbol{z},\Omega,\varepsilon,\mathcal{E}),\tilde{\Omega}(\Omega,\mathcal{E}))|\varepsilon,\mathcal{E},\boldsymbol{z},\Omega\right]$. To keep the notation short, we denote the two expectations terms by $\mathbb{E}_{-}\tilde{\boldsymbol{x}}$ and $\mathbb{E}_{+}\tilde{\boldsymbol{x}}$ respectively.[10] These conditions can be written as

$$F\left(\mathbb{E}_{-}\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}},\mathbb{E}_{+}\tilde{\boldsymbol{x}},\tilde{\boldsymbol{X}},\varepsilon,\mathcal{E},\boldsymbol{z}\right)=\boldsymbol{0} \tag{20}$$

for some mapping F. The remaining optimality conditions are various aggregate feasibility constraints and first-order conditions with respect to $\tilde{\boldsymbol{X}}$ that show the relationship between aggregate policy functions and integrals of individual policy functions. They can be written as

$$R\left(\int\tilde{\boldsymbol{x}}d\Omega,\tilde{\boldsymbol{X}},\mathcal{E}\right)=\boldsymbol{0} \tag{21}$$

for some mapping R. The law of motion for the distribution Ω is

$$\widetilde{\Omega}\left(\mathcal{E},\Omega\right)\left(\boldsymbol{z}\right)=\int\iota\left(\tilde{\boldsymbol{z}}\left(\varepsilon,\mathcal{E},\boldsymbol{y},\Omega\right)\leq\boldsymbol{z}\right)d\Pr\left(\varepsilon\right)d\Omega\left(\boldsymbol{y}\right)\quad\forall\boldsymbol{z}, \tag{22}$$

where $\iota\left(\tilde{\boldsymbol{z}}\leq\boldsymbol{z}\right)$ is 1 if all elements of $\tilde{\boldsymbol{z}}$ are less than all elements of $\boldsymbol{z}$, and zero otherwise. Equations (20)-(22) jointly determine the Ramsey allocation.

asset holdings).

[10] Strictly speaking, if $\tilde{\boldsymbol{x}}$ consists of all lower-case choice variables and multipliers in problem (19), then the relevant object is $\mathbb{E}_{-}f(\tilde{\boldsymbol{x}})$ and $\mathbb{E}_{+}g(\tilde{\boldsymbol{x}})$ for some transformations f and g. Our exposition is without loss of generality once we expand the definition of $\tilde{\boldsymbol{x}}$ to also include variables $f\left(\tilde{\boldsymbol{x}}\right)$ and $g\left(\tilde{\boldsymbol{x}}\right)$.

To approximate policy functions, we consider a family of economies parameterized by a positive scalar σ that scales all shocks $\varepsilon, \mathcal{E}$. We construct first-, second-, and higher-order approximations with respect to σ evaluated at $\sigma = 0$. Since, the parameter σ affects policy functions directly in addition to scaled shocks $(\sigma\varepsilon, \sigma\mathcal{E})$, we record it as an extra argument. The first-order expansion of $\tilde{\boldsymbol{X}}$ reads

$$\begin{aligned}\tilde{\boldsymbol{X}}(\sigma\mathcal{E}, \Omega; \sigma) &= \tilde{\boldsymbol{X}}(0, \Omega; 0) + \sigma\left(\tilde{\boldsymbol{X}}_{\mathcal{E}}(0, \Omega; 0)\mathcal{E} + \tilde{\boldsymbol{X}}_{\sigma}(0, \Omega; 0)\right) + \mathcal{O}(\sigma^2) \\ &\equiv \bar{\boldsymbol{X}} + \sigma\left(\bar{\boldsymbol{X}}_{\mathcal{E}}\mathcal{E} + \bar{\boldsymbol{X}}_{\sigma}\right) + \mathcal{O}(\sigma^2),\end{aligned} \tag{23}$$

where $\tilde{\boldsymbol{X}}_{\mathcal{E}}$ and $\tilde{\boldsymbol{X}}_{\sigma}$ are derivatives with respect to the first and third arguments, and bars indicate that functions are evaluated at $(0, \Omega; 0)$. The same convention applies to the law of motion $\widetilde{\Omega}(\sigma\mathcal{E}, \Omega; \sigma)$, e.g. $\bar{\Omega} = \widetilde{\Omega}(0, \Omega; 0)$. Individual policy functions $\tilde{\boldsymbol{x}}$ are expanded analogously as

$$\tilde{\boldsymbol{x}}(\sigma\varepsilon, \sigma\mathcal{E}, \boldsymbol{z}, \Omega; \sigma) = \bar{\boldsymbol{x}}(\boldsymbol{z}) + \sigma\left(\bar{\boldsymbol{x}}_{\varepsilon}(\boldsymbol{z})\varepsilon + \bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{z})\mathcal{E} + \bar{\boldsymbol{x}}_{\sigma}(\boldsymbol{z})\right) + \mathcal{O}(\sigma^2). \tag{24}$$

We describe next how to obtain the coefficients that appear in these expansions by using the implicit function theorem.

3.1.1 Points of expansion and zeroth-order terms

Our point of expansion $\sigma = 0$ is a deterministic economy. It is easy to see that the law of motion for distribution Ω_t is particularly simple in this case: if the economy starts with any given Ω_0, it remains with that distribution forever. Formally,

Lemma 1. *For any Ω, policy functions satisfy $\bar{\boldsymbol{z}}(\boldsymbol{z}) = \boldsymbol{z}$ for any $\boldsymbol{z}$ and therefore $\bar{\Omega}(\Omega) = \Omega$.*

The fact that the distribution remains stationary at $\sigma = 0$ simplifies our analysis. Although at first sight this result appears to be sensitive to the simplifying assumptions made in this section, we show later that by a careful choice of the individual state variables one can obtain an analogue of lemma 1 even if the $\sigma = 0$ economy features non-trivial deterministic dynamics.

Lemma 1 implies that the expectation of next period's functions is simply equal to their current period value at the point of expansion. Therefore, we can find $\bar{\boldsymbol{X}}$ and

$\bar{\boldsymbol{x}}(\boldsymbol{z})$ as a solution to a non-linear system of equations

$$F\left(\bar{\boldsymbol{x}}(\boldsymbol{z}), \bar{\boldsymbol{x}}(\boldsymbol{z}), \bar{\boldsymbol{x}}(\boldsymbol{z}), \bar{\boldsymbol{X}}, 0, 0, \boldsymbol{z}\right) = \mathbf{0}, \qquad R\left(\int \bar{\boldsymbol{x}}(\boldsymbol{z}) d\Omega(\boldsymbol{z}), \bar{\boldsymbol{X}}, 0\right) = \mathbf{0}.$$

This system can be solved efficiently using standard root finding algorithms.

Using the zeroth-order terms $\bar{\boldsymbol{X}}$ and $\bar{\boldsymbol{x}}(\boldsymbol{z})$, we construct several objects that will be used to find higher-order terms. Let $\mathsf{R}_{\boldsymbol{x}}$ and $\mathsf{R}_{\boldsymbol{X}}$ be the derivatives of the R mapping with respect to its first two arguments, and let $\mathsf{R}_{\mathcal{E}}$ be the derivative with respect to the scaled shock $\sigma\mathcal{E}$, all evaluated $\sigma = 0$. Similarly, let subscripts $\boldsymbol{x}-$, $\boldsymbol{x}$, $\boldsymbol{x}+$, $\boldsymbol{X}$, ε, $\mathcal{E}$ and $\boldsymbol{z}$ denote the analogous derivatives of F with respect to each of its arguments. From the implicit function theorem we have $\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}) = \left[\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\right]^{-1} \mathsf{F}_{\boldsymbol{z}}(\boldsymbol{z})$.

3.1.2 First-order terms

That aggregate shocks in period t affect the distributional state Ω in period $t+1$ makes calculating mathematical expectations of next period variables in equation (20) hard. A principal contribution of our approach is to simplify calculation of these expectations. The first step towards this goal is to use lemma 1 to show that

$$\begin{aligned}
\mathbb{E}\left[\tilde{\boldsymbol{x}}(\cdot,\cdot,\tilde{z}(\boldsymbol{z},\Omega,\varepsilon,\mathcal{E}),\tilde{\Omega}(\Omega,\mathcal{E}))|\varepsilon,\mathcal{E},\boldsymbol{z},\Omega\right] =& \bar{\boldsymbol{x}}(\boldsymbol{z}) + \left[\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{z}) + \partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\bar{\Omega}_{\mathcal{E}}\right]\sigma\mathcal{E} \\
&+ \left[\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}_{\varepsilon}(\boldsymbol{z})\right]\sigma\varepsilon + \bar{\boldsymbol{x}}_{\sigma}(\boldsymbol{z})\sigma + \mathcal{O}(\sigma^2), \\
\mathbb{E}\left[\tilde{\boldsymbol{x}}(\cdot,\cdot,z,\Omega)\right] =& \bar{\boldsymbol{x}}(\boldsymbol{z}) + \bar{\boldsymbol{x}}_{\sigma}(\boldsymbol{z})\sigma + \mathcal{O}(\sigma^2),
\end{aligned}$$

where $\partial\bar{\boldsymbol{x}}$ is the Frechet derivative of $\bar{\boldsymbol{x}}$ with respect to the distribution Ω.[11] The term $\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\bar{\Omega}_{\mathcal{E}}$ tells how current period's shock $\mathcal{E}$ influences the expectation of next period's policy function through its effect on next period's distribution Ω. Formally, $\bar{\Omega}_{\mathcal{E}}$ is the derivative of the law of motion (22) with respect to $\sigma\mathcal{E}$ evaluated at $\sigma = 0$ and $\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\bar{\Omega}_{\mathcal{E}}$ is the Frechet derivative $\partial\bar{\boldsymbol{x}}(\boldsymbol{z})$ evaluated at the point $\bar{\Omega}_{\mathcal{E}}$.

The derivative $\partial\bar{\boldsymbol{x}}(\cdot)$ is high dimensional. Calculating it explicitly would be infeasible when the state space is large. Luckily, we need to know the derivative only at

[11] A Frechet derivative $\partial\bar{\boldsymbol{x}}(\boldsymbol{z})$ is a linear operator from the space of distributions Ω to $\mathbb{R}$ with a property that $\lim_{\|\Delta\|\to 0}\frac{\|\bar{\boldsymbol{x}}(\boldsymbol{z},\Omega+\Delta)-\bar{\boldsymbol{x}}(\boldsymbol{z},\Omega)-\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta\|}{\|\Delta\|} = 0$. It can be found by fixing a feasible direction Δ and calculating a directional (Gateaux) derivative since $\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta = \lim_{\alpha\to 0}\frac{\bar{\boldsymbol{x}}(\boldsymbol{z},\Omega+\alpha\Delta)-\bar{\boldsymbol{x}}(\boldsymbol{z},\Omega)}{\alpha}$. See Chapter 7 of Luenberger (1997). Throughout, we assume that Δ vanishes on the boundaries of its domain, $\mathbb{R}_+ \times \mathbb{R}$.

the point $\bar{\Omega}_{\mathcal{E}}$. The next theorem shows that there exists a simple linear relationship between $\partial\bar{\boldsymbol{x}} \cdot \bar{\Omega}_{\mathcal{E}}$ and $\bar{\boldsymbol{x}}_{\mathcal{E}}$ that can be solved explicitly.

Theorem 1. ***(Factorization)*** *From the zeroth-order expansion one can construct matrices* $\mathsf{A}(\boldsymbol{z})$ *and* $\mathsf{C}(\boldsymbol{z})$ *such that*

$$\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) = \mathsf{C}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}, \tag{25a}$$

$$\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) \cdot \bar{\Omega}_{\mathcal{E}} = \mathsf{C}(\boldsymbol{z})\underbrace{\partial\bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\mathcal{E}}}_{\bar{\boldsymbol{X}}'_{\mathcal{E}}} = \mathsf{C}(\boldsymbol{z}) \int \mathsf{A}(\boldsymbol{y})\bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{y}) d\Omega\left(\boldsymbol{y}\right). \tag{25b}$$

Proof. Evaluate the Frechet derivatives of (20) and (21) at an arbitrary point Δ and use the fact that $\partial\bar{\Omega} = \mathbf{1}$ from lemma 1 to show that

$$\begin{aligned} \left(\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\right)\partial\bar{\boldsymbol{x}}(z) \cdot \Delta + \mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}} \cdot \Delta &= \mathbf{0}, & (26a) \\ \mathsf{R}_{\boldsymbol{x}}\partial\left(\int \bar{\boldsymbol{x}}\left(\boldsymbol{y}\right) d\Omega\left(\boldsymbol{y}\right)\right) \cdot \Delta + \mathsf{R}_{\boldsymbol{X}}\partial\bar{\boldsymbol{X}} \cdot \Delta &= \mathbf{0}. & (26b) \end{aligned}$$

The first equation yields (25a) with $\mathsf{C}(\boldsymbol{z}) = -\left(\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\right)^{-1}\mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})$.

By calculating directional derivatives in direction Δ we can show that

$$\partial\left(\int \bar{\boldsymbol{x}}\left(\boldsymbol{y}\right) d\Omega\left(\boldsymbol{y}\right)\right) \cdot \Delta = \int \left(\partial\bar{\boldsymbol{x}}\left(\boldsymbol{y}\right) \cdot \Delta\right) d\Omega\left(\boldsymbol{y}\right) + \int \bar{\boldsymbol{x}}\left(\boldsymbol{y}\right) d\Delta\left(\boldsymbol{y}\right). \tag{27}$$

We want to evaluate the integral on the right side at $\Delta = \bar{\Omega}_{\mathcal{E}}$. Differentiating (22) at any $\boldsymbol{z} = (m, \mu)$ and applying lemma 1, we show that

$$\bar{\Omega}_{\mathcal{E}}\left(m, \mu\right) = -\int_{y_2 \leq \mu} \bar{m}_{\mathcal{E}}\left(m, y_2\right) \omega\left(m, y_2\right) dy_2 - \int_{y_1 \leq m} \bar{\mu}_{\mathcal{E}}\left(y_1, \mu\right) \omega\left(y_1, \mu\right) dy_1,$$

where ω is the density of Ω. The density of $\bar{\Omega}_{\mathcal{E}}\left(m, \mu\right)$ is then

$$\bar{\omega}_{\mathcal{E}}\left(m, \mu\right) = -\frac{d}{dm}\left[\bar{m}_{\mathcal{E}}\left(m, \mu\right) \omega\left(m, \mu\right)\right] - \frac{d}{d\mu}\left[\bar{\mu}_{\mathcal{E}}\left(m, \mu\right) \omega\left(m, \mu\right)\right].$$

Substitute this equation and (25a) into (27) to get

$$\begin{aligned}\partial\left(\int \bar{\boldsymbol{x}}\left(\boldsymbol{y}\right)d\Omega\left(\boldsymbol{y}\right)\right)\cdot\bar{\Omega}_{\mathcal{E}} &= \int \mathsf{C}(\boldsymbol{y})\partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_{\mathcal{E}}d\Omega\left(\boldsymbol{y}\right) - \int \bar{\boldsymbol{x}}\left(\boldsymbol{y}\right)\frac{d}{dm}\left[\bar{m}_{\mathcal{E}}\left(\boldsymbol{y}\right)\omega\left(\boldsymbol{y}\right)\right]d\boldsymbol{y} \\ &\quad - \int \bar{\boldsymbol{x}}\left(\boldsymbol{y}\right)\frac{d}{d\mu}\left[\bar{\mu}_{\mathcal{E}}\left(\boldsymbol{y}\right)\omega\left(\boldsymbol{y}\right)\right]d\boldsymbol{y} \\ &= \int \mathsf{C}(\boldsymbol{y})\partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_{\mathcal{E}}d\Omega\left(\boldsymbol{y}\right) + \int \bar{\boldsymbol{x}}_{\boldsymbol{z}}\left(\boldsymbol{y}\right)\mathsf{p}\bar{\boldsymbol{x}}_{\mathcal{E}}\left(\boldsymbol{y}\right)d\Omega\left(\boldsymbol{y}\right),\end{aligned}$$

where the second equality is obtained using integration by parts. Substitute this expression into (26b) and solve for $\partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_{\mathcal{E}}$ to obtain

$$\bar{\boldsymbol{X}}'_{\mathcal{E}} \equiv \partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_{\mathcal{E}} = \int \mathsf{A}(\boldsymbol{y})\bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{y})d\Omega\left(\boldsymbol{y}\right), \tag{28}$$

where $\mathsf{A}(\boldsymbol{z}) = -\left(\mathsf{R}_{\boldsymbol{x}}\int\mathsf{C}(\boldsymbol{y})d\Omega\left(\boldsymbol{y}\right)+\mathsf{R}_{\boldsymbol{X}}\right)^{-1}\mathsf{R}_{\boldsymbol{x}}\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}$. Together with (25a) this proves (25b). □

The factorization theorem makes our approach computationally tractable when state space Ω is large. To see its significance, suppose that we approximate Ω on a grid with K points. In that case, the Frechet derivative of individual policy functions $\partial\bar{\boldsymbol{x}}(\cdot)$ contains K^2 unknown elements. Since the number of the unknowns grows exponentially with the number K of grid points, calculating this derivative directly would be infeasible for large K. Theorem 1 circumvents this problem by providing an explicit formula for the value of the derivative $\partial\bar{\boldsymbol{x}}(\cdot)$ at the point $\bar{\Omega}_{\mathcal{E}}$ as a linear function of (yet unknown) $\bar{\boldsymbol{x}}_{\mathcal{E}}\left(\boldsymbol{z}\right)$ weighted with coefficients given by matrices $\mathsf{A}(\boldsymbol{z}), \mathsf{C}(\boldsymbol{z})$. Matrices $\mathsf{A}(\boldsymbol{z}), \mathsf{C}(\boldsymbol{z})$ can be calculated quickly from the zeroth-order terms since the only non-linear operation required is inversion of matrices of dimension no greater than $\max\left\{\dim\boldsymbol{x}, \dim\boldsymbol{X}\right\}$. Because $\mathsf{A}(\boldsymbol{z}), \mathsf{C}(\boldsymbol{z})$ can be calculated for different values of $\boldsymbol{z}$ independently, the algorithm is easy to parallelize, a good thing when K is very large.

An economic intuition underlies derivations in the proof of theorem 1. In competitive equilibrium agents care about the distribution Ω only to the extent it affects aggregate prices and income. Thus, the effect from any perturbation of distribution Ω on individual variables, $\partial\bar{\boldsymbol{x}}$, can be factorized into the effect of that perturbation on aggregate variables, $\partial\bar{\boldsymbol{X}}$, and a known loading matrix C that captures how individual variables respond to changes in the aggregates. Equation (25a) captures this relationship. Feasibility and market clearing constraints impose a tight relationship between individual policy function responses to aggregate shocks in the current period, $\bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{z})$, and expected future changes in the aggregates, $\bar{\boldsymbol{X}}'_{\mathcal{E}}$. Conceptually, this relationship is the a fixed point problem and we derive an explicit solution in equation (28). Together with (25a), this allows us to find $\partial\bar{\boldsymbol{x}}\cdot\bar{\Omega}_{\mathcal{E}}$ in terms of $\bar{\boldsymbol{x}}_{\mathcal{E}}$ without having to compute $\partial\bar{\boldsymbol{x}}(\cdot)$.

We can now take the first-order expansion of (20) and (21), apply the factorization theorem, and use the method of undetermined coefficients to find $\bar{\boldsymbol{X}}_{\mathcal{E}}, \bar{\boldsymbol{X}}_{\sigma}, \bar{\boldsymbol{x}}_{\mathcal{E}}, \bar{\boldsymbol{x}}_{\varepsilon}, \bar{\boldsymbol{x}}_{\sigma}$. We illustrate how this is done for $\bar{\boldsymbol{X}}_{\mathcal{E}}$, $\bar{\boldsymbol{x}}_{\mathcal{E}}$. Consider the first-order expansion of the left hand side of equations (20) and (21). Because this expansion should be equal to $\mathbf{0}$ for all values of $\mathcal{E}$, the matrices of coefficients that multiply $\mathcal{E}$ should equal $\mathbf{0}$ as well. Solving for these matrices explicitly, we obtain a system of equations

$$\left(\mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\right)\bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\mathsf{C}(\boldsymbol{z})\bar{\boldsymbol{X}}'_{\mathcal{E}} + \mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\mathcal{E}} + \mathsf{F}_{\mathcal{E}}(\boldsymbol{z}) = \mathbf{0}, \tag{29a}$$

$$\mathsf{R}_{\boldsymbol{x}}\int \bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{y})d\Omega\left(\boldsymbol{y}\right) + \mathsf{R}_{\boldsymbol{X}}\bar{\boldsymbol{X}}_{\mathcal{E}} + \mathsf{R}_{\mathcal{E}} = \mathbf{0}. \tag{29b}$$

This is a linear system with unknowns $\bar{\boldsymbol{X}}_{\mathcal{E}}, \bar{\boldsymbol{x}}_{\mathcal{E}}$ after we substitute the definition of $\bar{\boldsymbol{X}}'_{\mathcal{E}}$ from (28). Solving the discretized version directly would require inverting a square matrix of dimension $K\dim\boldsymbol{x}$, which is impractical when the dimension of the grid (K) is large. A more computationally efficient approach is to split this system into K independent systems of dimensions $\dim\boldsymbol{x}$ that can be solved in parallel. To this end, use equation (29a) to calculate matrices $\mathsf{D}_0(\boldsymbol{z})$ and $\mathsf{D}_1(\boldsymbol{z})$ that define the linear relationship

$$\bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{z}) = \mathsf{D}_0(\boldsymbol{z}) + \mathsf{D}_1(\boldsymbol{z})\cdot\left[\begin{array}{cc}\bar{\boldsymbol{X}}_{\mathcal{E}} & \bar{\boldsymbol{X}}'_{\mathcal{E}}\end{array}\right]^{\mathsf{T}}.$$

Then use this relationship to substitute into equations (29b) and (28) to find $\bar{\boldsymbol{X}}_{\mathcal{E}}$. Values of $\bar{\boldsymbol{x}}_{\mathcal{E}}(\boldsymbol{z})$ can be found either by substituting back into the previous equation or from (25a).

The remaining unknown first-order coefficients $\bar{\boldsymbol{X}}_{\sigma}, \bar{\boldsymbol{x}}_{\varepsilon}, \bar{\boldsymbol{x}}_{\sigma}$ can be found using similar steps. The steps for computing them are much simpler since they do not depend on the Frechet derivative $\partial\bar{\boldsymbol{x}}$.

3.1.3 Second- and higher-order expansions

Our approach extends to second- and higher-order expansions while preserving the computationally convenient linear structure. The key insight is that the factorization theorem generalizes to higher order expansions. The analogue of equation (25a) holds for any order of perturbations of Ω. This allows us to solve for higher order analogues of $\partial\bar{\boldsymbol{x}} \cdot \bar{\Omega}_{\mathcal{E}}$ explicitly as weighted sums of higher-order coefficients $\bar{\boldsymbol{x}}_{\mathcal{E}\mathcal{E}}$, $\bar{\boldsymbol{x}}_{\mathcal{E}\sigma}$, $\bar{\boldsymbol{x}}_{\sigma\sigma}$ ..., with weights known from lower-order expansions. We then can form higher order analogues of the system of equations (29). As before, the mathematical structure of these equations allows us to split one large system of equations into a large number of low dimensional linear problems that can be solved fast and simultaneously. Formal proofs and constructions are notation-intensive but the steps mirror those in section 3.1.2 and we confine them to the online appendix.

3.2 Approximations in the general case

We now turn to the economy described in section 2. For concreteness we focus on the case when $\ln \Theta_t$ follows an AR(1) process (9a). This economy has two differences from the section 3.1 simpler economy. First, since shocks are persistent, policies also depend on previous period values $\boldsymbol{\Theta} = (\Theta, \Phi)$ and θ. Second, the Philips curve constraint (15) generally binds and its Lagrange multiplier Λ becomes a state variable. Thus, in the general economy $\boldsymbol{z} = (m, \mu, s, \theta)$ is the individual state, Ω is a distribution over this four-dimensional state space, and the aggregate and individual policy functions are mappings $\tilde{\boldsymbol{X}}(\boldsymbol{\mathcal{E}}, \Omega, \boldsymbol{\Theta}, \Lambda)$ and $\tilde{\boldsymbol{x}}(\boldsymbol{\mathcal{E}}, \boldsymbol{\varepsilon}, \boldsymbol{z}, \Omega, \boldsymbol{\Theta}, \Lambda)$ respectively.

To make progress, we assume that persistence of idiosyncratic shocks ρ_θ is close to 1 so that we can represent it as $\rho_\theta = 1 - \sigma\rho$ for some $\rho \geq 0$. We follow the approach of the previous section by considering a sequence of economies with scaled shocks $\sigma\boldsymbol{\mathcal{E}}$ and $\sigma\boldsymbol{\varepsilon}$ and persistence $1 - \sigma\rho$. We take their first- and second-order expansion with respect to σ evaluated at $\sigma = 0$ around any given distribution Ω. Let $\bar{X}(\boldsymbol{\Theta}, \Lambda)$ and $\bar{x}(\boldsymbol{z}, \boldsymbol{\Theta}, \Lambda)$ be their zeroth-order values.

Our general economy can have non-trivial dynamics even when $\sigma = 0$. This would present a difficulty if the complicated part of the state space – the distribution Ω – also changed over time. A judicious choice of state variables allows us to avoid this complication.

Lemma 2. *For any* Ω*, policy functions satisfy* $\bar{\boldsymbol{z}}(\boldsymbol{z},\boldsymbol{\Theta},\Lambda)=\boldsymbol{z}$ *for any* $(\boldsymbol{z},\boldsymbol{\Theta},\Lambda)$ *and therefore* $\bar{\Omega}(\boldsymbol{\Theta},\Lambda)=\Omega$ *for any* $\boldsymbol{\Theta},\Lambda$*.*

The intuition is as follows. There are multiple mathematically equivalent ways to choose individual state variables. We chose them to be Pareto-Negishi weights m and multipliers on implementability constraints μ. In the deterministic economy (or, more generally, in a complete market economy as in Lucas and Stokey (1983) or Werning (2007)) these weights are constant over time, even when other choices of state variables, such as agents' debt levels, would not be.

Lemma 2 paves a way for the natural extension of our algorithm from section 3.1. Start with any $(\boldsymbol{\Theta}_0,\Lambda_0)$ and compute the deterministic transition dynamics as $\{\boldsymbol{\Theta}_t,\Lambda_t\}_t$ converge to the steady state. The steady state value of $\boldsymbol{\Theta}$ is simply $\bar{\boldsymbol{\Theta}}=\left(\bar{\Theta},\bar{\Phi}\right)$, while the steady state value of Λ, that we denote by $\bar{\Lambda}$, is pinned down by a single non-linear equation $\tilde{\Lambda}\left(\mathbf{0},\Omega,\bar{\boldsymbol{\Theta}},\bar{\Lambda};0\right)=\bar{\Lambda}$. Theorem 1 and its higher-order generalizations continue to hold along the transition path of $\{\boldsymbol{\Theta}_t,\Lambda_t\}_t$, allowing one to solve for the coefficients in expansions of policy functions using backward induction, as in Anderson et al. (2012). This procedure can be further simplified by simply expanding $\tilde{\boldsymbol{x}}(\boldsymbol{\mathcal{E}},\boldsymbol{\varepsilon},\boldsymbol{z},\Omega,\boldsymbol{\Theta},\Lambda)$ and $\tilde{\boldsymbol{X}}(\boldsymbol{\mathcal{E}},\Omega,\boldsymbol{\Theta},\Lambda)$ around $\boldsymbol{\Theta}=\bar{\boldsymbol{\Theta}}$ and $\Lambda=\bar{\Lambda}$ rather than solving for the full transition dynamics $\{\boldsymbol{\Theta}_t,\Lambda_t\}_t$. The online appendix describes how implement both approaches.

4 Calibration and computations

We choose three sets of parameters: (i) parameters governing preferences, technology and aggregate shocks; (ii) initial conditions; and (iii) stochastic processes for idiosyncratic shocks.

Preferences, technology, aggregate shocks

Our settings of aggregate parameters align with standard representative agent calibrations such as Schmitt-Grohe and Uribe (2004) and Siu (2004). We set utility function parameters at $\nu=1$, $\gamma=2$, and the discount factor β at 0.96. We assume that firms operate a constant returns to scale technology and set $\alpha=1$. To a first approximation, average markups are $\left(\bar{\Phi}-1\right)^{-1}$ and the slope of the Phillips curve is $\left(\bar{\Phi}-1\right)/\psi$. We set $\bar{\Phi}=6$ to attain average markups of 20% and $\psi=20$ to match the slope of the Phillips curve as estimated by Sbordone (2002).[12] We set government non-transfer expenditures $\bar{G}$ at 0.09 in order to match the average federal government current expenditures net of transfer payments to annual GDP from the NIPA for the period 1947-2016.

The stochastic process for the markup shocks is calibrated to be consistent with that estimated in Smets and Wouters (2007) with the AR coefficient of 0.65 and the volatility of the innovation to $\ln \Phi$ such that a one standard deviation negative shock $\mathcal{E}_\Phi$ raises firms' optimal markups by about 0.5 percentage points. We set the standard deviation of $\mathcal{E}_\Theta$ at 3% to approximate the process for output per worker in the U.S. In the AR(1) specification of TFP shocks we set $\rho_\Theta = 0.8$ and $\bar{\Theta} = 1$.

Initial conditions

We use the 2007 wave of the Survey of Consumer Finances (SCF) to calibrate the initial distribution of wages, debt, and equity holdings. We restrict our sample to married households who work at least 100 hours. To measure bond holdings, we sum direct and indirect holdings of government bonds through mutual funds (taxable and nontaxable), saving bonds, liquid assets (net of unsecured credit), money market accounts, and components of retirement accounts that are invested in government bonds. To measure equity holdings we sum direct holdings of equities and indirect holdings through mutual funds and retirement accounts.[13]

To fit initial states $\{\theta_{i,-1}, b_{i,-1}, s_i\}$ to data, we sample directly from the SCF log wages, debt and shares of equity using 2007 wave sampling weights. The SCF provides population weights for each observation. Given these weights, we set the initial condition by drawing with replacement a random sample of 100000 agents from a discrete distribution. In our SCF sample, 30% of households hold zero equities; the distribution of equities among the remaining households is right skewed with the top 10% of agents holding about 60% of total equities. Debt holdings, equity holdings, and wages are positively correlated. Table I reports summary statistics of the data and our fit.

[12] Using quarterly data, Sbordone (2002) estimates the slope of the U.S. Phillips curve to be about 0.06. Since our calibration uses annual frequency, we multiply her number by 4.

[13] We drop observations where equity or bond holdings are more than 100 times average yearly wage. These turned out to be about 0.5% of the total sample.

TABLE I: FIT OF THE INITIAL DISTRIBUTION

	Data	Model
fraction of pop. with zero equities	30%	30%
std. share of equities	2.62	2.61
std. bond	4.90	5.23
std. ln wages	0.81	0.82
corr(share of equities,ln wages)	0.40	0.39
corr(share of equities,bond holdings)	0.59	0.55
corr(bond,ln wages)	0.32	0.30

Notes: The moments in the data column corresponds to SCF 2007 wave with sample restrictions explained in the text and after scaling wages, equity holdings, and debt holdings by the average yearly wage in our sample. The share of equities refers to the ratio of individual equity holdings to the total in our sample such that the weighted sum of shares equals one.

Stochastic process for idiosyncratic shocks

We set the stochastic process for idiosyncratic shocks to match some facts about labor earnings reported by Storesletten et al. (2004) and Guvenen et al. (2014). In the model, labor earnings depend both on the stochastic process for skill $\epsilon_{i,t}$ and on monetary-fiscal policy. We calibrate them to fit a competitive equilibrium in which interest rates and tax rates are set to match stylized features of U.S. policies, namely,

$$Q_t^{-1} - 1 = \frac{1}{\beta} + 1.5\Pi_t, \qquad \Upsilon_t = 0.25. \tag{30}$$

This Taylor rule parameterization for nominal rates is common in the literature, while the calibrated tax rate matches estimates of the federal average marginal income tax rate reported by Barro and Redlick (2011), who also show that it is largely acyclical at business cycle frequencies.

To calibrate parameters governing stochastic processes (10) and (11), we simulate log earnings for 100000 agents using a competitive equilibrium with the initial condition and government policies as described above. Shocks $\varepsilon_{\epsilon,i,t}$, $\varepsilon_{\theta,i,t}$ are Gaussian. Storesletten et al. (2004) estimates a transitory/persistent component model of log earnings using the PSID. We targeted their estimates by setting the parameter ρ_θ to match the autocorrelation of the persistent component of log earnings, parameters $std(\varepsilon_{\epsilon,i,t})$, $std(\varepsilon_{\theta,i,t})$ to match the standard deviation of the transitory and persistent

TABLE II: PARAMETERS FOR IDIOSYNCRATIC SHOCKS PROCESSES

Parameters	Values	Targeted moment	Values	Source
s.d of ε_ϵ	0.17	s.d. of transitory component in log earnings	0.25	Storesletten et al. (2004)
s.d. of ε_θ	0.12	s.d. of persistent component in log earnings	0.12	
ρ_θ	0.99	autocorr of persistent component in log earnings	0.99	
f_2, f_1, f_0	0.28,−0.52, 0.00	relative (to median) earnings losses $5^{th}, 50^{th}$ & 95^{th} percentiles	see text	Guvenen et al. (2014), see figure I

Notes: The parameters in this table characterize the stochastic processes in equations (10) and (11).

components of log earnings.[14]

Guvenen et al. (2014) report earnings losses in the four recessions between 1978-2010 by percentiles of 5 year averages of pre-recession earnings. These provide us with moments that we use to construct the loading function $f(\theta)$. We assume a quadratic form $f(\theta) = f_0 + f_1\theta + f_2\theta^2$ and calibrate $\{f_0, f_1, f_2\}$ as follows. We first normalize Guvenen et al. (2014) reported earning losses for each percentile by the median earnings losses and average across all four recessions to obtain a profile of relative (to median) earnings losses. This is the dashed line in figure I. We next simulate the competitive equilibrium for 50 periods with a recession that is ignited by one standard deviation negative TFP shock. Following the empirical procedure in Guvenen et al. (2014), we rank workers by percentiles of their average log labor earnings 5 years prior to the shock and compute the percent earnings loss for each percentile relative to the median. The parameters f_0, f_1, f_2 are set to match earnings losses of the 5^{th}, 50^{th} and 95^{th} percentiles. Our model's counterpart to Guvenen et al. (2014) is the solid line in figure I. Parameters governing the process for idiosyncratic shocks are summarized in table II.

[14]In the model, we compute these cross-sectional moments by simulating a panel of log earnings from the competitive equilibrium with 100000 agents after turning off aggregate shocks for a sample of length 50 periods.

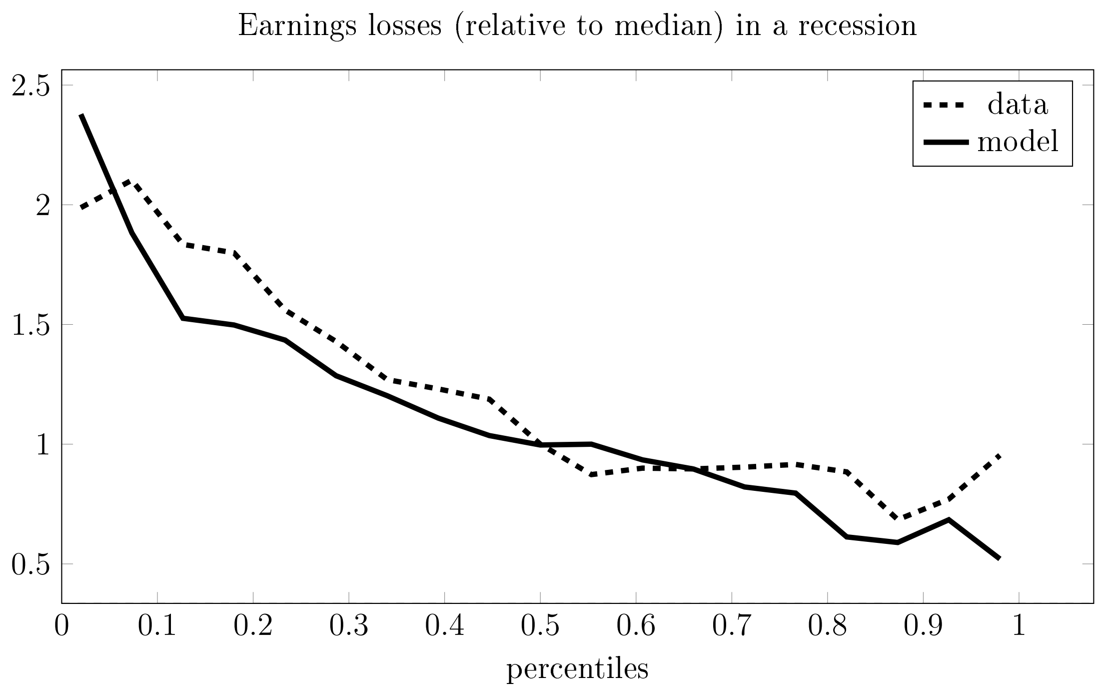

Figure I: Relative (to median) earnings losses by percentiles of 5 year average pre-recession log earnings. The data line is constructed using moments from Guvenen et al. (2014) who report the profile of earnings losses in the last 4 recessions (1978-2010). In the model line, we use simulated earnings before and after a one standard deviation TFP shock.

Computations

We use our second-order expansion formulas to approximate Ramsey policies. To compute the optimal monetary policy, we set $\bar{\Upsilon}$ to the optimal level in the non-stochastic environment with the same the initial condition. At every history we discretize the distribution across agents with $K = 2000$ grid points that we choose each period using a k-means clustering algorithm. We apply our algorithm to approximate optimal policies, use Monte Carlo simulation to draw idiosyncratic shocks and finally compute the aggregate distribution next period for a given history of aggregate shocks. The algorithm is then repeated with the new distribution. The details of how to implement the simulation steps are in the online appendix.

We focus on optimal policy responses to aggregate shocks $\mathcal{E}_{\Phi,t}, \mathcal{E}_{\Theta,t}$. These responses depend on the underlying state. To compute them we draw 100 histories of shocks $\{\mathcal{E}_{\Phi,t}, \mathcal{E}_{\Theta,t}\}_{t=0}^{25}$. To calculate an impulse response, say, to productivity shock $\mathcal{E}_{\Theta,k}$, of size δ in period k, we create two replicas of these draws. In one replica we replace $\mathcal{E}_{\Theta,k}$ from 0 and in the other with δ. We then integrate over all histories to obtain expected paths conditional on $\mathcal{E}_{\Theta,k} = 0$ and on $\mathcal{E}_{\Theta,k} = \delta$. Our impulse responses are the differences in the two paths. Since those paths are the same by construction for all $t < k$, we draw all the graphs starting with $t = k$. For concreteness we report all responses for $k = 5$.[15]

5 Results

We study optimal monetary and monetary-fiscal responses to one standard deviation negative shocks to $\mathcal{E}_{\Phi,t}$ and $\mathcal{E}_{\Theta,t}$. These responses are primarily shaped by two goals. The first is price stability, which is also present in representative agent New Keynesian models. Price changes are costly and by pursuing stable prices the planner can mitigate these costs. The second is provision of insurance, a goal that arises from heterogeneity and market incompleteness because aggregate shocks affect different agents differently. If agents could trade a complete set of Arrow securities, they would insure these shocks away and maintain constant consumption shares. Our

[15] As often occurs (for example, see Lucas and Stokey (1983), Schmitt-Grohe and Uribe (2004)) Ramsey policy at time $t = 0$ differs from continuation Ramsey policies for $t > 1$, being functions of different arguments. In our economy, by $t = 5$, transient effects of time 0 actions have mostly dissipated.

assumptions rule out those trades. The planner uses monetary and fiscal policies to make the distribution of gains and losses more equal.

In what follows we compare optimal responses in our economy to those in a representative agent version that shuts down all idiosyncratic shocks and heterogeneity across agents. The RANK economy provides a useful benchmark because it isolates the role of price stability concerns. We refer to our baseline setting with heterogeneous agents as HANK and the representative agent counterpart as RANK. When we study Ramsey monetary policy, for each economy we fix $\bar{\Upsilon}$ at the associated optimal level in the non-stochastic environment. In the RANK economy it is $\bar{\Upsilon} = -1/\bar{\Phi}$.

5.1 Optimal Responses to Markup Shocks

We start by considering the optimal response to a negative innovation in $\mathcal{E}_{\Phi,t}$. As this shock increases the desired markup $1/(\Phi_t - 1)$, we refer to it as a positive markup shock. Figure II shows (solid line) that the planner responds to a 0.5 percentage points increase in markups by cutting the nominal interest rate by 0.3 percentage points. This response boosts output, inflation and real wages. The optimal policy response to a markup shock differs significantly from that in the representative agent model (dashed line).

To understand economic forces behind these results, consider implications of the shock for price stability and insurance. A negative innovation to $\mathcal{E}_{\Phi,t}$ increases firms' desired markups over marginal costs which, *ceteris paribus*, makes firms want to increase nominal prices. To reduce the cost of price changes the planner can aim to lower marginal costs by increasing the nominal interest rate, thereby depressing aggregate demand and real wages, which reduces marginal costs. Dubbed "leaning against the wind" by Galí (2015), this response is optimal in representative agent New Keynesian economies. The interest rate increase that maintains stable prices is modest (see the dashed line).

Now turn to the insurance considerations. Higher markups increase dividends and lower the real wage. This benefits agents who own a lot of equities and hurts agents who mainly rely on labor income. To provide insurance that offsets the differential impact of the markup shock, the planner can aim to increase the real wage by cutting the nominal interest rate and boosting aggregate demand.

Thus, the Ramsey planner's concerns about price stability and insurance have

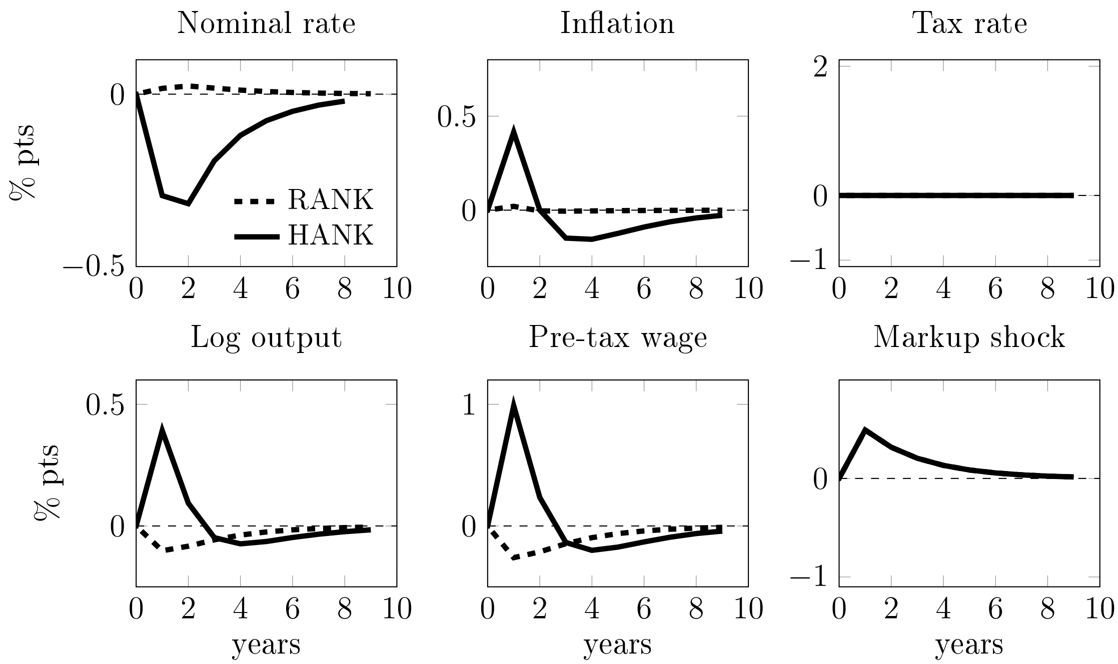

Figure II: Optimal monetary response to a markup shock

opposite implications for an optimal monetary response. Figure II shows that quantitatively the planner's intention to provide insurance swamps the intention to stabilize prices. To understand why the insurance motive is so strong, think about how the distribution of gains and losses from the shock depends on asset inequality. The insurance motive is least active when firm ownership is widely dispersed among agents and most active when ownership is very skewed. Data portray the distribution of stock market wealth as highly unequal, much more than the distribution of labor earnings. That makes the insurance motive relatively strong.

Figure III shows optimal monetary-fiscal responses to a markup shock. The planner temporarily increases tax rates and lowers nominal interest rates. This combination of policies increases the pre-tax wage W and inflation.

Labor taxes allow the planner to completely offset the inflationary consequences of the markup shock. If taxes are decreased to make the path of $1-\tau$ mirror the path of the shock, then the after-tax wage $(1-\tau)W$ and inflation are both unchanged. Since this tax response offsets the effect of the shock on inflation, the planner need not change the nominal interest rate. In representative agent New Keynesian economies (dashed line) this policy achieves the first best.

In our economy, this fiscal response to an aggregate shock damages insurance pro-

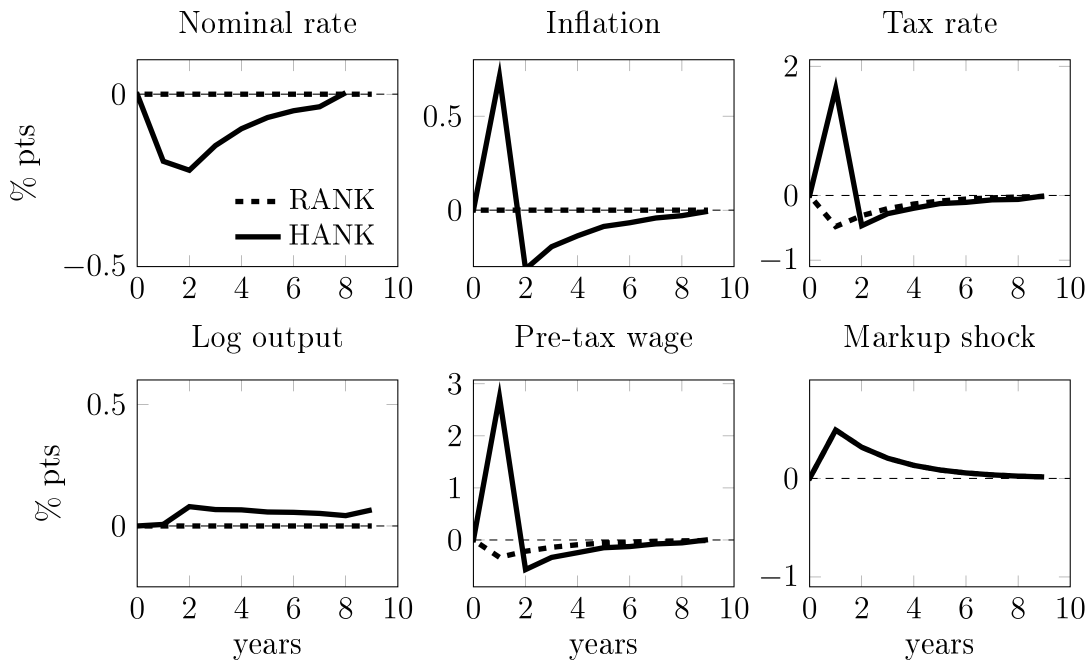

Figure III: Optimal monetary-fiscal response to a markup shock

vision. When tax rates are lowered, the planner has to decrease transfers in order to satisfy the government's budget constraint. That makes average taxes more regressive and amplifies the adverse effect of the markup shock on low-wage households. To provide insurance, the planner wants to make the tax system more progressive by increasing transfers funded by higher marginal tax rates. It is optimal to combine this fiscal policy with loose monetary policy because that offsets the negative effect of tax rate increases on output. Quantitatively, the insurance considerations dominate the optimal fiscal response.

In contrast to tax-smoothing prescriptions of Barro (1979) and Aiyagari et al. (2002), the optimal tax rate increase is short-lived. This is because transitory tax changes are less distortionary than permanent changes when price are sticky. A permanent increase in tax rates has no effect on real pre-tax wages W as in the long run firms set their prices as a constant markup over wages. As a result, a permanent tax rate change increases the labor distortion $(1-\tau)W$ one for one. In contrast, firms find it suboptimal to adjust prices fully to a transient tax rate increase. Thus, a transitory tax rate increase raises the pre-tax wage and has little (zero, in a continuous time limit) effect on the labor distortion.

5.2 Optimal Responses to Productivity Shocks

Price stability and insurance provision also shape Ramsey responses to productivity shocks. A well-known prescription for price stability is to move nominal interest rates one for one with changes in the "natural rate of interest" – the real interest rate that would prevail without nominal frictions. Consumers' Euler equations then ensure that inflation remains unchanged in response to the shock. The growth rate specification for productivity shocks in (9b) keeps the natural rate constant providing us with a clean way to isolate the insurance motive.

Figure IV shows the optimal monetary response to a negative innovation in $\mathcal{E}_{\Theta,t}$. This response is driven purely by insurance concerns: in the RANK economy where these concerns are absent, the planner optimally keeps interest rates and inflation unchanged. Several economic forces explain why lower aggregate productivity generates insurance motives in our model. One is heterogeneity in bond holdings. Incomes of agents who hold a lot of non-state-contingent bonds suffer less from an adverse productivity shock than incomes of agents who are in debt. This effect is present even if wages of all agents are the same. Two features in the data that we target amplify this differential effect: (i) low wage workers typically have fewer assets, and (ii) the direct exposures of labor earnings to aggregate productivity shocks are large for low wage workers, captured by the shape of our loading function f.

To provide insurance, the planner wants to transfer resources from high income to low income agents. Since debt is nominal, that can be done by lowering ex-post real returns on debt after a negative $\mathcal{E}_{\Theta,t}$ shock. A lower return makes the distribution of losses between savers and borrowers more equal. The planner lowers realized real returns on the nominal asset by cutting nominal interest rates, thereby increasing inflation. The policy response is sizable but short lived.[16]

Figure V shows an optimal monetary-fiscal response to an aggregate productivity shock. When the planner has access to fiscal policy, increasing transfers funded by higher labor tax rates provides insurance against the adverse productivity shock. First, more progressive average taxes redistribute labor income towards low-wage, low-asset agents. This by itself would mandate a persistent increase in tax rates

[16] Recall that we follow the usual convention in the New Keynesian literature and assume that debt matures in one period. With longer debt maturity, the decreases in nominal interest rates and increases in inflation are smaller but more persistent which allows the planner to lower the cost of nominal price adjustments.

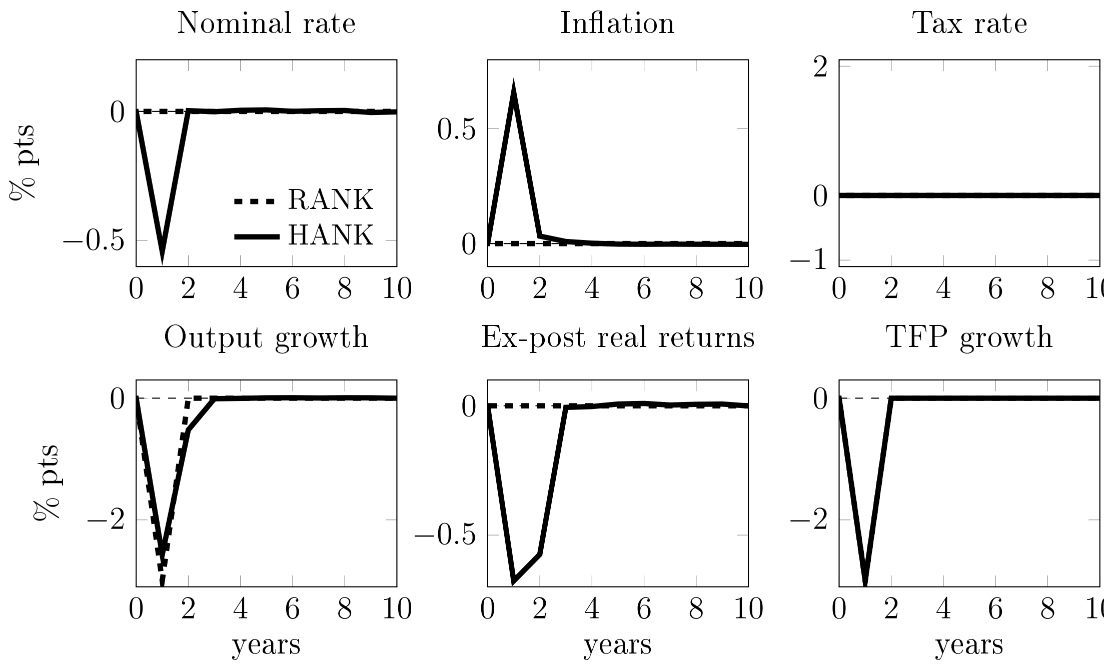

Figure IV: Optimal monetary response to a productivity growth rate shock

because the increase in inequality is long lived. But a second force works through changes in the pre-tax wage. With costly price adjustments, a temporary spike in marginal taxes rates raises real wages. Together these two effects explain the responses of optimal tax rates in figure V. Although nominal interest rates move little, that behavior masks a substantial departure from natural interest rate targeting. The response of optimal tax rates increases the natural rate in the first period, putting the nominal rate substantially below it. The difference between nominal and natural interest rates follows a path that is similar to that in figure IV. This explains the spike in inflation that provides insurance through lower realized returns on bond holdings.

Now consider the alternative AR(1) specification of productivity shocks in equation (9a). A negative shock $\mathcal{E}_{\Theta,t}$ predicts a higher growth rate of Θ_{t+k} in the future. This raises the natural rate, so in order to attain price stability the planner would want to raise the nominal interest rate as well. This is the main difference from the growth rate specification of productivity shocks. In particular, once we look at the difference between nominal and natural rates instead of the nominal interest rate itself, counterparts of figures IV and V are very similar in both cases. We report them in the online appendix where we also explain how to extend the notion of the natural rate to our HANK economy.

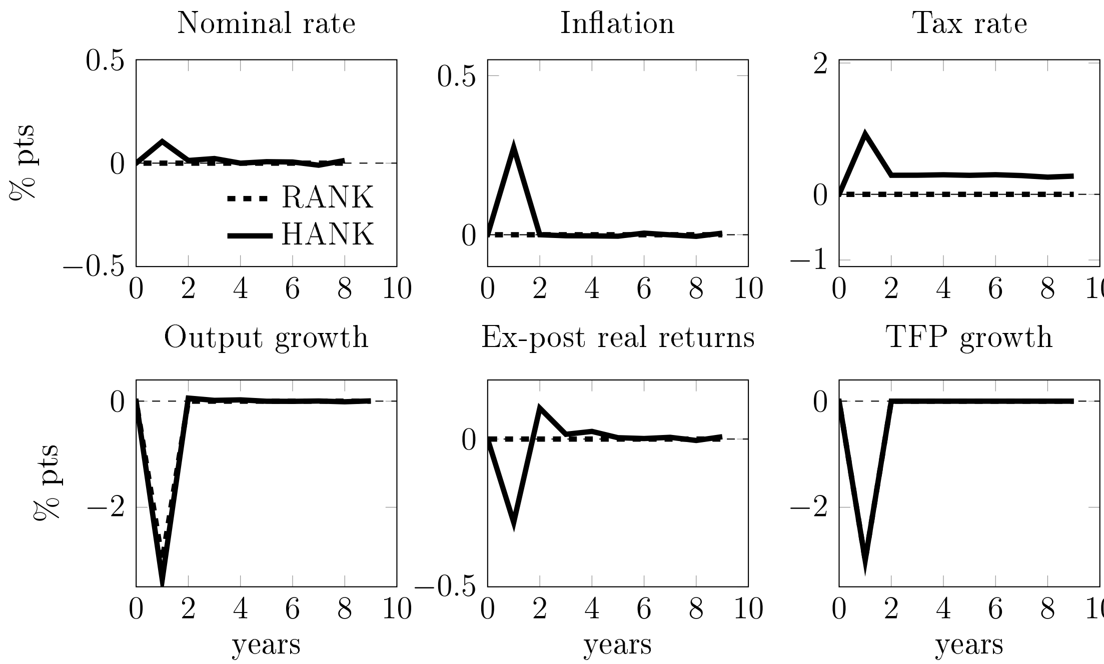

Figure V: Optimal monetary-fiscal response to a productivity growth rate shock

5.3 Taylor Rules

We explore how well a Taylor rule approximates the Ramsey plan. To this end we fix tax rates as in sections 5.1 and 5.2 but assume that monetary policy follows the interest rule specified in equation (30).

Figure VI compares responses of interest rates and inflation under the Ramsey plan and the Taylor rule. The left panel (black lines) in figure VI shows that the responses implied by the Taylor rules are too small, too persistent, and, in the case of markup shocks, are in the opposite direction of the optimal responses. We could have anticipated that a Taylor rule does a poor job of approximating the Ramsey responses. In sections 5.1 and 5.2, we emphasized that the main economic force driving optimal policy is the need to provide insurance; Taylor rules are designed to achieve price stability. In the right panel (red lines) in figure VI we plot responses to markup and TFP shocks in the representative agent economy and find that qualitatively, the interest rate and inflation both behave similarly under Ramsey and Taylor policies.[17]

[17] In the RANK economy it easy to improve the fit further by small changes in the coefficients in the Taylor rule. For example, if we set the coefficient on inflation to 5, the two responses lie almost perfectly on top of each other. This is not surprising and previous studies, for instance Woodford (2003) and Galí (2015), also find that Taylor rules approximate the optimal Ramsey policies well in

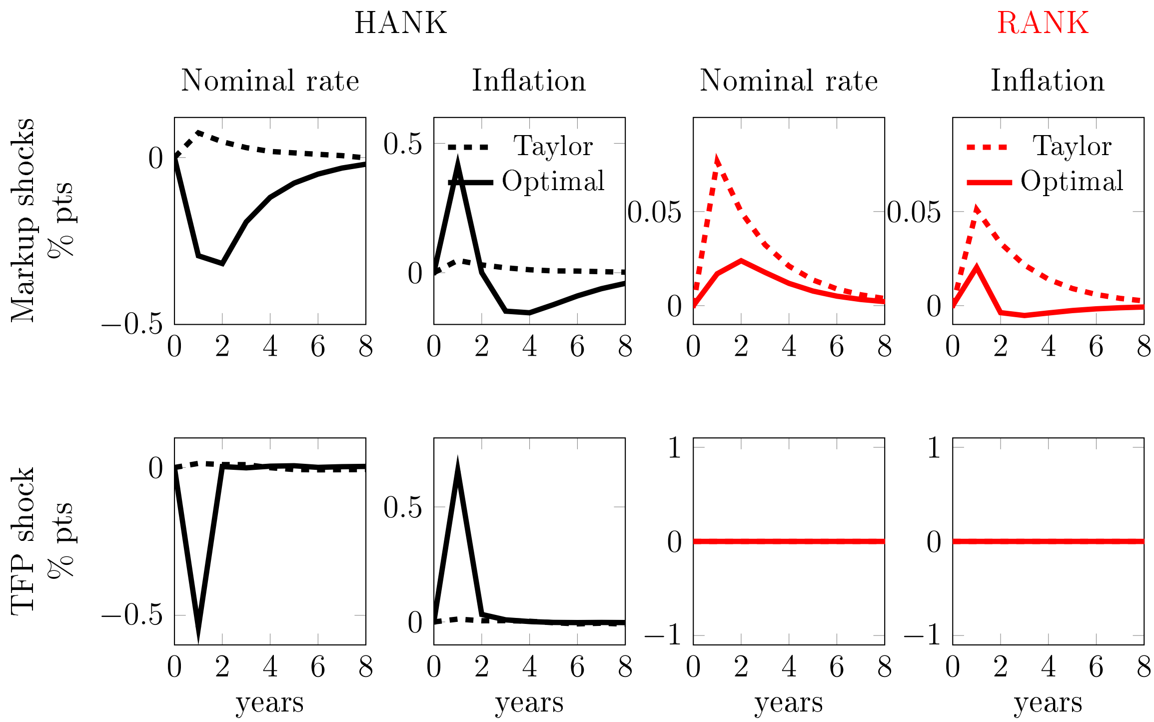

Figure VI: Comparing optimal monetary responses to Taylor rule in HANK (black, left panel) and RANK (red, right panel) models. The solid line is the optimal response and the dashed line is the response in a competitive equilibrium with $i_t = \bar{i} + 1.5\pi_t$. For TFP shocks we use the growth rate specification (9b).

The difficulty with a Taylor rule lies not only in its conventional parameterization but in a restriction that it implicitly imposes on the persistence of interest rates and inflation. Under a Taylor rule, persistence of nominal interest rates is inherited from the persistence of exogenous aggregate shocks. Thus, a long-lived markup shock implies long-lived changes in nominal rates and inflation. A motive to provide insurance in the HANK economy often requires transient and short-lived changes in interest rate and inflation. These are difficult to implement with standard Taylor rules.

6 Extensions and robustness

We consider several extensions to the section 2 environment.

a class of representative agent settings.

6.1 Heterogeneity in the marginal propensity to consume

Our baseline specification combines a New Keynesian framework with a standard Bewley-Aiyagari model of idiosyncratic shocks and imperfect insurance. A well known limitation of Bewley-Aiyagari models is that they fail to generate sufficient heterogeneity in marginal propensities to consume (MPCs).[18] That prediction is at odds with the data. For example, Jappelli and Pistaferri (2014) document large differences in MPCs across agents and show that MPCs systematically co-vary with wealth. Recent studies (Auclert (2017), Kaplan et al. (2018)) stress the role of MPC heterogeneity in transmitting changes in interest rates. We are interested in implications of MPC heterogeneity for *optimal* monetary policy.

We follow Jappelli and Pistaferri (2014) and augment our model with "hand-to-mouth" agents. These agents can own equities and consume dividends (in addition to labor income) but cannot trade financial assets over time. Thus, we modify our baseline set up to allow two dimensions of permanent heterogeneity, equity holding s_i and a new variable h_i, where $h_i \in \{0, 1\}$ indicates whether or not individual i is a hand-to-mouth type. We assume that the probability of being hand-to-mouth is

$$\Pr(h_i = 1|s_i) = a_0 + a_1 \times \text{Percentile}\,(s_i),$$

and re-calibrate our model by choosing parameters a_0 and a_1 to match the MPC gradient with respect to wealth reported in Jappelli and Pistaferri (2014) in addition to our calibration targets in section 4. Table III shows Jappelli and Pistaferri (2014) numbers and the goodness of fit of our calibration.

This approach is in line with Kaplan et al. (2018) who model MPC heterogeneity using a Bewley-Aiyagari model with costly adjustment between liquid and illiquid assets. Their baseline calibration makes MPC higher for agents with low liquid wealth, and about two thirds of agents with zero liquid wealth have positive illiquid wealth. Our specification captures both features, with high-s hand-to-mouth agents behaving like wealthy hand-to-mouth agents in Kaplan et al. (2018).

Frictions in trading financial assets segment agents into those who are unconstrained and are on their Euler equation and those who are constrained and at strict

[18]In our baseline model agents trade subject to the natural borrowing limit. The same conclusion also holds for economies when agents face ad-hoc borrowing limits, except for a small mass of agents near that limit. Krusell and Smith (1998) refer to this property in such models as "approximate aggregation."

TABLE III: MODEL FIT FOR MPC HETEROGENEITY

Parameters	Values	Moment	Data	Model
a_0	-0.35	MPC for 10 percentile of Cash on Hand	0.62	0.63
a_1	0.75	MPC for 90 percentile of Cash on Hand	0.35	0.33

Notes: The data column is obtained from figure 2 of Jappelli and Pistaferri (2014) who report average MPC out of transitory income shock by percentiles of cash on hand (CoH) where CoH is defined as the sum of household disposable income and financial wealth, net of consumer debt. In the model we compute CoH as the sum post - tax wage earnings, holdings of debt and equities.

Euler inequalities. Ramsey plans are still determined by price stability and insurance concerns but now they take the different policy responses of these two groups into account. In a setting with trading frictions, the effectiveness of monetary policy depends on the mechanism through which nominal rates provide insurance. To see this we compare the responses to markup and TFP shocks.

In the top row of figure VII we see that in response to a positive markup shock the planner still decreases the nominal interest rate, but not by as much as in the baseline model. Recall from section 5.1 that, by stimulating aggregate demand, cuts in interest rates provide insurance after a markup shock. Changes in interest rates directly affect only agents who are on their Euler equation, and so trading frictions diminish the ability of monetary policy to influence aggregate demand. In contrast, the bottom row of figure VII shows that the response of interest rates to a TFP shock is virtually unaffected by trading frictions. In this case, the planner provides insurance by lowering real returns on nominal debt. Asset prices are determined only by a marginal investor. Thus, in spite of trading frictions, monetary policy behaves similarly to the benchmark case.

Figure VII also shows the importance of transfers in providing insurance when agents face frictions in trading assets. In our baseline economy agents borrow subject to natural debt limits, so Ricardian equivalence holds and the timing of transfers is irrelevant. When some agents are constrained in their abilities to trade financial assets, transfers become an effective tool for affecting aggregate demand and real wages. For example, in response to a markup shock the planner increases transfers to stimulate aggregate demand. This leads to a spike in inflation of about the same magnitude as in the baseline economy despite a much smaller cut in nominal interest

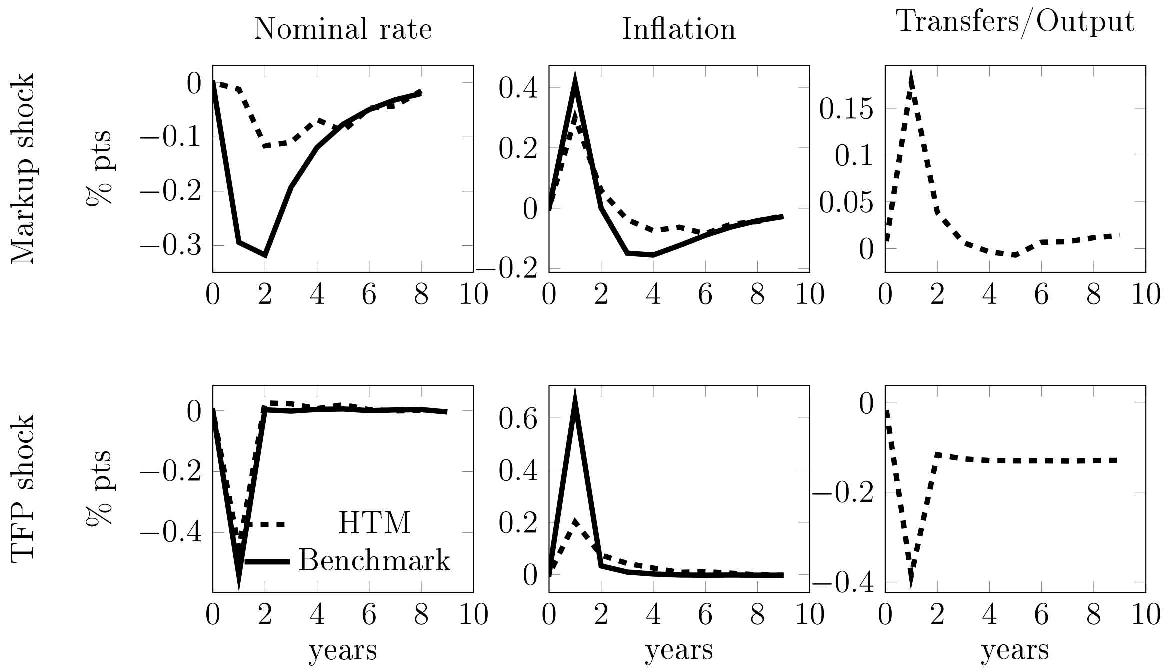

Figure VII: Optimal monetary responses with hand-to-mouth agents.

rates. Transfers are cut in response to a TFP shock because lower tax revenues make it harder for the government to finance its outstanding liabilities.

6.2 Alternative assumptions about welfare criterion and fiscal instruments

We consider other extensions of our baseline specification where we alter the welfare criteria; turn off heterogeneous exposures of TFP shocks by setting $f = 0$; and turn off nominal rigidities by setting $\psi = 0$. We summarize main insights from these extensions here and more details are in the online appendix.

In our baseline specification we assumed that the Ramsey planner is utilitarian meaning that he puts equal Pareto weights on all agents. This assumption is not essential for our results. The focus of our analysis has been on the role of monetary policy in providing insurance against aggregate shocks, and the need for this insurance is driven by market incompleteness not Pareto weights. To confirm this insight we tried a range of alternative Pareto weights, for example weights that rationalize the observed average marginal tax rates or alternative weights that prefer more redistribution. Different choices of Pareto weights have a big influence on the *average* levels of optimal tax rates and transfers but have little effect on how tax rates, transfers, and nominal interest rates respond to aggregate shocks.

We explored how assumptions about the menu of available fiscal instruments affect conclusions. Some of heterogeneous impacts of recessions in the data are driven by unemployment. Targeted instruments such as unemployment insurance (UI) can be more effective than nominal interest rates in helping agents to smooth their consumption. But even the best designed programs are likely to provide only partial insurance, due, for example, to moral hazard. Modeling these effects realistically would take us too far from the focus of this paper. One way to estimate a rough upper bound on the effectiveness of UI programs is to assume that they fully eliminate all heterogeneous wage impacts by setting $f = 0$. In this case, we find that the optimal response to a markup shock is essentially unaffected and that the response to a productivity shock is cut by about 50 percent. Unemployment insurance is not effective at providing insurance against shocks that affect wage earners and financial asset holders differently.

We also study the sensitivity of our findings to the size of nominal rigidities. Monetary policy remains effective even if prices are completely flexible because it still can influence the nominal price level and real returns on nominal debt. When we set $\psi = 0$ we find that inflation responds by 0.5% to markup shocks and by 2.5% to productivity shocks. These effects are larger than the baseline case where $\psi = 20$. The insight that nominal rigidities lower the optimal response of inflation is consistent with RANK studies (Schmitt-Grohe and Uribe (2004)), but unlike those papers the volatility of nominal interest rates and inflation remains sizable under conventional calibrations of nominal rigidities.

7 Concluding Remarks

James Tobin described macroeconomics as a field that explains aggregate quantities and prices while ignoring distribution effects. Tobin's characterization describes much work about real business cycles, asset pricing, Ramsey tax and debt, and New Keynesian models. In each of these lines of research, an assumption of complete markets and/or of a representative consumer allows the analyst to compute aggregate quantities and prices without also determining distributions across agents. This paper departs from Tobin's "aggregative economics" by assuming heterogeneous agents and incomplete markets.

Our paper makes two contributions. It develops a way to approximate optimal plans in economies with heterogeneous agents. Our method can be more broadly applied to settings that feature incomplete markets, idiosyncratic and aggregate shocks. Our quantitative application reevaluates lessons for monetary and fiscal policy drawn from New Keynesian economies. Relative to price stability motives that typically drive policy prescriptions, heterogeneity adds a quantitatively important insurance motive. Our findings complement a message in Kaplan and Violante (2018 forthcoming) that monetary policy is inextricably connected with fiscal policy and about distributional effects.

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Online Appendix

A Recursive Representation of Ramsey Plan

For completeness and ease of readability, we repeat the planning problem with a full set of implementability constraints below. Given an initial condition $\{b_{i,-1}, \theta_{i,-1}, s_i\}$, $\Lambda_{-1} = 0$, $a_{i,-1} = 0$ and the multipliers, the Ramsey allocation solves:

$$\begin{aligned}
\inf\sup \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \int \Bigg[\left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right) + \left(c_{i,t}^{1-\nu} - c_{i,t}^{-\nu}(T_t + s_i D_t) - n_{i,t}^{1+\gamma} \right) \mu_{i,t} \\
- \frac{a_{i,t-1} c_{i,t}^{-\nu} (1+\Pi_t)^{-1} \xi_{i,t}}{\beta \mathbb{E}_{t-1}\left[c_{i,t}^{-\nu} (1+\Pi_t)^{-1} \right]} + (1-\beta) c_{i,0}^{-\nu} b_{i,-1} \mu_{i,0} di \\
+ \Lambda_t C_t^{-\nu} Y_t \left[1 - \Phi_t \left(1 - \frac{W_t}{\alpha N_t^{\alpha-1}} \right) \right] + (\Lambda_{t-1} - \Lambda_t) C_t^{-\nu} \psi \Pi_t (1+\Pi_t) \Bigg\},
\end{aligned} \tag{31}$$

subject to

$$\begin{aligned}
c_{i,0}^{-\nu}(1-\Upsilon_0) W_0 \epsilon_{i,0} &= n_{i,0}^{\gamma} \\
m_{i,0}^{-1/\nu} c_{i,0} &= C_0 \\
\int c_{i,0} di &= C_0 \\
C_0 + \bar{G} &= N_0^{\alpha} - \frac{\psi}{2} \Pi_0^2 \\
D_0 &= N_0^{\alpha} - W_0 N_0 - \frac{\psi}{2} \Pi_0^2 \\
N_0 &= \int \epsilon_{i,0} n_{i,0} di
\end{aligned}$$

and

$$c_{i,t}^{-\nu}W_t(1-\Upsilon_t)\epsilon_{i,t} = n_{i,t}^{\gamma} \tag{32}$$

$$\mathcal{Q}_{t-1} = \beta m_{i,t-1}\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\Pi_{t+1})^{-1}\right] \tag{33}$$

$$m_{i,t}^{-1/\nu}c_{i,t} = C_t \tag{34}$$

$$\int c_{i,t}di = C_t \tag{35}$$

$$C_t + \bar{G} = N_t^{\alpha} - \frac{\psi}{2}\Pi_t^2 \tag{36}$$

$$D_t = N_t^{\alpha} - W_tN_t - \frac{\psi}{2}\Pi_t^2 \tag{37}$$

$$N_t = \int \epsilon_{i,t}n_{i,t}di \tag{38}$$

$$\mu_{i,t} = \mu_{i,t-1} + \xi_{i,t}.$$

for $t \geq 1$, where $\mathcal{Q}_{t-1} \equiv C_{t-1}^{-\nu}Q_{t-1}$, with

$$\begin{aligned}
\ln \epsilon_{i,t} &= \ln \Theta_t + \ln \theta_{i,t} + \varepsilon_{\epsilon,i,t} \\
\ln \theta_{i,t} &= \rho_\theta \ln \theta_{i,t-1} + f(\theta_{i,t-1})\mathcal{E}_{\Theta,t} + \varepsilon_{\theta,i,t} \\
\ln \Theta_t &= \rho_\Theta \ln \Theta_{t-1} + (1-\rho_\Theta)\ln \bar{\Theta} + \mathcal{E}_{\Theta,t} \\
\ln \Phi_t &= \rho_\Phi \ln \Phi_{t-1} + (1-\rho_\Phi)\ln \bar{\Phi} + \mathcal{E}_{\Phi,t},
\end{aligned}$$

where the inf is over multipliers $\{\mu_{i,t}, \Lambda_t\}_t$ and the sup is over $\left\{\{c_{i,t}, n_{i,t}, b_{i,t}\}_i, C_t, N_t, B_t, W_t, P_t, Y_t, D_t\right\}$.

The state for the $t \geq 1$ continuation problem is the joint distribution of $(m_{i,0}, \mu_{i,0}, \theta_{i,0}, s_i)$, Λ_0, and aggregate shocks (Θ_0, Φ_0). For the remainder of this appendix we'll focus on the first-order conditions for the continuation problem and how to approximate solutions to those first-order conditions recursive in the distribution over $(m_i, \mu_i, \theta_i, s_i)$ and (Λ, Θ, Φ). Once that approximation has been obtained, the solution to the time 0 problem can be found by solving the time 0 first-order conditions with the knowledge that some variables will be given by the continuation policy rules. We omit those first-order conditions for brevity.

Let $\beta^t\phi_{i,t}, \beta^t\rho_{i,t-1}, \beta^t\varphi_{i,t}, \beta^t\chi_t, \beta^t\Xi_t, \beta^t\zeta_t, \beta^t\kappa_t$ be the Lagrange multipliers on (32)-(38) respectively. After taking derivatives, the first-order condition with respect to $a_{i,t-1}$ combined with $\mu_{i,t} = \mu_{i,t-1} + \xi_{i,t}$ imply

$$\mu_{i,t-1} = \frac{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\Pi_t)^{-1}\mu_{i,t}\right]}{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\Pi_t)^{-1}\right]},$$

and the first-order conditions with respect to $m_{i,t}$ and $\mathcal{Q}_{t-1}$ give

$$\int \phi_{i,t} = 0.$$

Due to the timing of the planner's problem (at time t but before shocks have been realized) policy functions will be a functions of both the previous log of the shocks, $\boldsymbol{\Theta}_{t-1} = (\ln \Theta_{t-1}, \ln \Phi_{t-1})$; the innovations, $\boldsymbol{\mathcal{E}}_t = (\mathcal{E}_{\Theta,t}, \mathcal{E}_{\Phi,t})$ and $\boldsymbol{\varepsilon} = (\varepsilon_\epsilon, \varepsilon_\theta)$; as well as the individual states, $\boldsymbol{z}_{t-1} = (m_{t-1}, \mu_{t-1}, \theta_{t-1}, s)$, and the distribution over those states, Ω_{t-1}. The first order conditions of the planner's problem can then be written recursively as

$$\begin{aligned}
0 =& \tilde{n}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{1+\gamma} + \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu}(\tilde{T}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) + s\tilde{D}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)) + \tilde{a}(\boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\frac{\tilde{p}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)}{\beta \mathbb{E}\left[\tilde{p}(\cdot, \cdot, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\right]} \\
&- c(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{1-\nu} - \tilde{a}(\rho_{\boldsymbol{\Theta}}\boldsymbol{\Theta} + (1-\rho_{\boldsymbol{\Theta}})\bar{\boldsymbol{\Theta}} + \boldsymbol{\mathcal{E}}, \tilde{\boldsymbol{z}}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda), \tilde{\Omega}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda), \tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)) \qquad (39)
\end{aligned}$$

$$0 = \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu}\tilde{W}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)(1 - \tilde{\Upsilon}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))\tilde{\epsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \tilde{n}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{\gamma} \qquad (40)$$

$$0 = \tilde{m}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-1/\nu}\tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \qquad (41)$$

$$\begin{aligned}
0 =& \beta \mathbb{E}\left[\tilde{\varrho}(\cdot, \cdot \rho_{\boldsymbol{\Theta}}\boldsymbol{\Theta} + (1-\rho_{\boldsymbol{\Theta}})\bar{\boldsymbol{\Theta}} + \boldsymbol{\mathcal{E}}, \tilde{\boldsymbol{z}}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda), \tilde{\Omega}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda), \tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))\right] \\
&+ \frac{1}{\nu}\tilde{m}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-1/\nu-1}\tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\tilde{\varphi}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) \qquad (42)
\end{aligned}$$

$$\begin{aligned}
0 =& \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu} + (1-\nu)\tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu}\tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \beta\nu\tilde{m}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\frac{\tilde{\varrho}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)}{\tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)} \\
&- \nu\tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu-1}\left(\tilde{T}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) + s(\boldsymbol{z})\tilde{D}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\right)\tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) \\
&+ \frac{\nu\tilde{a}(\boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\tilde{r}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)}{\tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)}(\tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \mu) \\
&+ \nu\tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu-1}\tilde{W}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)(1 - \tilde{\Upsilon}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))\tilde{\epsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\tilde{\phi}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) \\
&+ \tilde{m}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-1/\nu}\tilde{\varphi}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \tilde{\chi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \qquad (43)
\end{aligned}$$

$$\begin{aligned}
0 =& \tilde{n}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{\gamma} - (1+\gamma)\tilde{n}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{\gamma}\tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) \\
&+ \gamma\tilde{n}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{\gamma-1}\tilde{\phi}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) + \tilde{\epsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\tilde{\kappa}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \qquad (44)
\end{aligned}$$

$$0 = \tilde{\varrho}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu}(1 + \tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))^{-1}\tilde{\rho}(\boldsymbol{z}, \Omega, \Lambda) \qquad (45)$$

$$0 = \tilde{p}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu}(1 + \tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))^{-1} \qquad (46)$$

$$0 = \tilde{r}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \frac{\tilde{p}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)}{\beta \mathbb{E}\left[\tilde{p}(\cdot, \cdot, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\right]} \qquad (47)$$

$$0 = \mu - \beta \mathbb{E}\left[\tilde{r}(\cdot, \cdot, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\right] \qquad (48)$$

$$0 = \tilde{\mathcal{Q}}(\boldsymbol{\Theta}, \Omega, \Lambda) - \beta m \mathbb{E}\left[\tilde{p}(\cdot, \cdot, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda))^{-1}\right] \qquad (49)$$

$$0 = \ln\tilde{\epsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \ln\tilde{\theta}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \varepsilon_\epsilon - \rho_\Theta \ln\Theta - (1-\rho_\theta)\ln\bar{\Theta} - \mathcal{E}_\Theta \qquad (50)$$

$$0 = \ln\tilde{\theta}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \rho_\Theta \ln\theta - f(\theta) - \varepsilon_\theta$$

$$0 = \tilde{s}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - s \qquad (51)$$

while the aggregate constraints are

$$0 = \int\int \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu} \tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) d\Pr(\boldsymbol{\varepsilon}) d\Omega(\boldsymbol{z}) \tag{52}$$

$$0 = \tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \int\int \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) d\Pr(\boldsymbol{\varepsilon}) d\Omega(\boldsymbol{z}) \tag{53}$$

$$0 = \tilde{Y}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \frac{\psi}{2}\tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^2 - \tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \bar{G} \tag{54}$$

$$0 = \int\int \tilde{\epsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) \tilde{n}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) d\Pr(\boldsymbol{\varepsilon}) d\Omega(\boldsymbol{z}) - \tilde{Y}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \tag{55}$$

$$0 = \tilde{Y}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \tilde{W}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{N}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \frac{\psi}{2}\tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^2 - \tilde{D}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \tag{56}$$

$$\begin{aligned} 0 =& \tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{-\nu}\tilde{Y}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\left(\left[1 - \tilde{\Phi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)(1 - \frac{1}{\alpha}\tilde{W}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{N}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{1-\alpha})\right]\right) \\ & - \tilde{\aleph}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) + \beta\mathbb{E}\left[\tilde{\aleph}(\cdot, \rho_{\boldsymbol{\Theta}}\boldsymbol{\Theta} + (1-\rho_{\boldsymbol{\Theta}})\bar{\boldsymbol{\Theta}} + \boldsymbol{\mathcal{E}}, \tilde{\Omega}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda), \tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))\right] \end{aligned} \tag{57}$$

$$\begin{aligned} 0 =& \tilde{\chi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \tilde{\Xi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) + \nu\frac{\tilde{\aleph}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)}{\tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)}(\tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \Lambda) \\ & - \nu\tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{-\nu-1}\left(\tilde{Y}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\left[1 - \tilde{\Phi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)(1 - \tilde{W}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))\right]\right)\tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \end{aligned} \tag{58}$$

$$\begin{aligned} 0 =& (1 + \tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))^{-1}\int\int \tilde{a}(\boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\tilde{r}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)(\tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \mu) \\ & - \beta\tilde{\varrho}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) d\Pr(\boldsymbol{\varepsilon}) d\Omega(\boldsymbol{z}) - \psi\tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{\Xi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \end{aligned} \tag{59}$$

$$- \psi\tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{\zeta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \psi\tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{-\nu}(1 + 2\tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda))(\tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \Lambda) \tag{60}$$

$$0 = \int\int \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu}\tilde{\epsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)\tilde{\phi}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) d\Pr(\boldsymbol{\varepsilon}) d\Omega(\boldsymbol{z}) \tag{61}$$

$$0 = \int \tilde{c}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda)^{-\nu} s\tilde{\mu}(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) d\Pr(\boldsymbol{\varepsilon}) d\Omega - \tilde{\zeta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \tag{62}$$

$$0 = \frac{1}{\alpha}\tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{-\nu}\tilde{\Phi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \tilde{\zeta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \tag{63}$$

$$0 = \tilde{\Lambda}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\left(\alpha\tilde{N}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{\alpha-1}(1 - \tilde{\Phi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)) + \frac{1}{\alpha}\tilde{\Phi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{W}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\right) \tag{64}$$

$$+ \alpha\tilde{N}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{\alpha-1}\left(\tilde{\Xi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) + \tilde{\zeta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\right) - \tilde{W}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)\tilde{\zeta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \tilde{\kappa}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) \tag{65}$$

$$0 = \tilde{\aleph}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \tilde{C}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{-\nu}\psi\tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)(1 + \tilde{\Pi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)) \tag{66}$$

$$0 = \tilde{Y}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \tilde{N}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)^{\alpha} \tag{67}$$

$$0 = \tilde{\Phi}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda) - \exp(\rho_{\Phi}\ln\Phi + (1-\rho_{\Phi})\ln\bar{\Phi} + \mathcal{E}_{\Phi}) \tag{68}$$

$$0 = \int \tilde{\rho}(\boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) d\Omega \tag{69}$$

$$0 = \mathbb{E}\left[\tilde{T}(\cdot, \boldsymbol{\Theta}, \Omega, \Lambda)\right] \tag{70}$$

These are in addition to the implicit restrictions that $\tilde{Q}, \tilde{a}$ and $\tilde{\rho}$ do not depend on period t shocks. The additional constraint (70) is required because of Ricardian equivalence as without it there would be multiple solutions to the FOC corresponding to different paths of government debt.

A.1 Slack Phillips Curve When $s_i = 1$

Here we demonstrate the result that when all agents have the same equity ownership, $s_i = 1$

for all i, the Phillips curve, equation (15), does not bind at the optimal allocation and hence $\Lambda_t = 0$ for all t. Let $\{c_{i,t}, n_{i,t}\}$ and $\{C_t, \Pi_t, W_t, T_t, \Upsilon_t, D_t, Q_t, Y_t\}$ be the allocation that maximizes a relaxed planner's problem without the Phillip's curve constraint. Define $\hat{W}_t$ such that (15) is satisfied; $\hat{D}_t = (1-\hat{W}_t)Y_t - \frac{\psi}{2}\Pi_t^2$; $\hat{\Upsilon}_t$ such that $(1-\hat{\Upsilon}_t)\hat{W}_t = (1-\Upsilon_t)W_t$; and $\hat{T}_t$ such that $\hat{T}_t + s_i\hat{D}_t = T_t + s_i D_t$. It is clear from this definition that all equilibrium conditions will then be satisfied with $\{C_t, \Pi_t, \hat{W}_t, \hat{T}_t, \hat{\Upsilon}_t, \hat{D}_t, Q_t, Y_t\}$ for the allocation $\{c_{i,t}, n_{i,t}\}$, and therefore this allocation maximizes the full planner's problem which implies that the Phillip's curve is slack.

B Additional Details for Section 3

This section contains the omitted steps and proofs from section 3 in the main text. In section B.1 we define the F and R functions used in the expansion provided in section B.2. We prove lemma 1 and lemma 2 of the main text in section B.3. In sections B.4-B.7, we derive equations (74)-(79) which extend the factorization theorem, theorem 1, to the more general Ramsey problem, and then exploit this factorization to derive equations (76)-(96) which give all the derivatives needed for the second-order Taylor expansions of the Ramsey policies using matrices known from the lower-order expansions and requiring only small dimensional matrix inversions.

B.1 Deriving functions F and R

Let $\tilde{\boldsymbol{x}} = \left(\tilde{m}, \tilde{\mu}, \tilde{\theta}, \tilde{s}, \tilde{c}, \tilde{n}, \tilde{a}, \tilde{\varrho}, \tilde{\phi}, \tilde{\varphi}, \tilde{\rho}, \tilde{r}, \tilde{p}, \tilde{\epsilon}\right)^{\mathsf{T}}$ denote the household specific variables and let $\tilde{\boldsymbol{X}} = \left(\tilde{\Lambda}, \tilde{C}, \tilde{Y}, \tilde{\Pi}, \tilde{W}, \tilde{D}, \tilde{N}, \tilde{\aleph}, \tilde{\chi}, \tilde{\Xi}, \tilde{\Phi}, \tilde{\zeta}, \tilde{\kappa}, \tilde{T}, \tilde{\Upsilon}\right)^{\mathsf{T}}$ denote the aggregate variables. By assumption, $\tilde{\Lambda}$ is contained in $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{z}}$ is contained in $\tilde{\boldsymbol{x}}$ so there exist projection matrices P and p that satisfy $\tilde{\Lambda} = \mathsf{P}\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{z}} = \mathsf{p}\tilde{\boldsymbol{x}}$. Equations (39)-(51) represent the individual constraints of the problem and can be summarized by the following function which extends (20) of section 3

$$0 = F\Bigg(\boldsymbol{z}, \mathbb{E}\tilde{\boldsymbol{x}}(\cdot,\cdot,\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda), \tilde{\boldsymbol{x}}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda),$$
$$\mathbb{E}\left[\tilde{\boldsymbol{x}}(\cdot,\cdot,\rho_{\boldsymbol{\Theta}}\boldsymbol{\Theta} + (1-\rho_{\boldsymbol{\Theta}})\bar{\boldsymbol{\Theta}} + \boldsymbol{\mathcal{E}}, \tilde{\boldsymbol{z}}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda), \tilde{\Omega}(\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda), \tilde{\Lambda}(\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\right], \tag{71}$$
$$\tilde{\boldsymbol{X}}(\boldsymbol{\Theta},\Omega,\Lambda), \boldsymbol{\varepsilon}, \mathcal{E}, \boldsymbol{\Theta}\Bigg)$$

and must hold for all $\boldsymbol{z}$ in the support of Ω. Equations (52)-(70) capture the aggregate constraints which must hold across all agents and can be defined by the following function which extends (21) of section 3[19]

$$0 = \int R\Bigg(\boldsymbol{z}, \tilde{\boldsymbol{x}}(\boldsymbol{\varepsilon}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda), \tilde{\boldsymbol{X}}(\boldsymbol{\Theta}, \Omega, \Lambda), \\ \mathbb{E}[\tilde{\boldsymbol{X}}(\cdot, \rho_{\boldsymbol{\Theta}}\boldsymbol{\Theta} + (1-\rho_{\boldsymbol{\Theta}})\bar{\boldsymbol{\Theta}} + \boldsymbol{\mathcal{E}}, \tilde{\Omega}(\boldsymbol{\Theta}, \Omega, \Lambda), \tilde{\Lambda}(\boldsymbol{\Theta}, \Omega, \Lambda))], \boldsymbol{\varepsilon}, \boldsymbol{\Theta}, \Lambda \Bigg) d\Pr(\boldsymbol{\varepsilon}) d\Omega. \tag{72}$$

The law of motion for $\tilde{\Omega}$ is given by

$$\widetilde{\Omega}\left(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda\right)(y) = \int \iota\left(\tilde{z}\left(\varepsilon, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda\right) \leq y\right) d\Pr\left(\varepsilon\right) d\Omega \quad \forall y, \tag{73}$$

where $\iota\left(\tilde{\boldsymbol{z}}\left(\varepsilon, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda\right) \leq y\right)$ is 1 iff all elements of $\tilde{\boldsymbol{z}}\left(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega\right)$ are less than all elements of y and zero otherwise.

B.2 Setting up the expansions

Section 3 presented two approaches to approximating $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{X}}$: expanding around the deterministic transition dynamics $\{\boldsymbol{\Theta}_t, \Lambda_t\}$ and expanding around the non-stochastic steady state $(\bar{\boldsymbol{\Theta}}, \bar{\Lambda})$. We first present how to expand around the non-stochastic steady state $(\bar{\boldsymbol{\Theta}}, \bar{\Lambda})$ in sections B.2-B.6. In section B.7 we describe the full expansion along the transition path. To expand around $(\bar{\boldsymbol{\Theta}}, \bar{\Lambda})$, we consider a positive scalar σ that scales all shocks, deviations of $\boldsymbol{\Theta}$ and Λ from their respective steady states $\bar{\boldsymbol{\Theta}}$ and $\bar{\Lambda}(\Omega)$, and $\rho_\theta(\sigma) = 1 - \sigma\rho$. We let $\tilde{\boldsymbol{x}}(\sigma\boldsymbol{\varepsilon}, \sigma\boldsymbol{\mathcal{E}}, \sigma(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) + \bar{\boldsymbol{\Theta}}, \boldsymbol{z}, \Omega, \sigma(\Lambda - \bar{\Lambda}(\Omega)) + \bar{\Lambda}; \sigma)$ and $\tilde{\boldsymbol{X}}(\sigma\boldsymbol{\mathcal{E}}, \sigma(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) + \bar{\boldsymbol{\Theta}}, \boldsymbol{z}, \Omega, \sigma(\Lambda - \bar{\Lambda}(\Omega)); \sigma)$ denote the policy rules with scaling parameters σ, and approximate these policies with a Taylor expansion with respect to σ. For example, a first-order Taylor expansion would be of the form

$$\tilde{\boldsymbol{x}}(\boldsymbol{\varepsilon}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) = \bar{\boldsymbol{x}}(\boldsymbol{z}, \Omega) + \left(\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}, \Omega)\boldsymbol{\varepsilon} + \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}, \Omega)\boldsymbol{\mathcal{E}} + \bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z}, \Omega)(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}})\right)\sigma \\ + \left(\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z}, \Omega)(\Lambda - \bar{\Lambda}(\Omega)) + \bar{\boldsymbol{x}}_\sigma\right)\sigma + \mathcal{O}(\sigma^2)$$

[19]The R presented here nests that of section 3 with two main differences. The first is the inclusion of expectations of future aggregate variables. The second is that we choose to integrate over R rather than over x. Both choices are without loss of generality but they allow for equations (39)- (51) and (52)-(70) representing the Ramsey plan to be expressed in a more condensed form.

and

$$\tilde{\boldsymbol{X}}(\boldsymbol{\Theta},\Omega,\Lambda)=\bar{\boldsymbol{X}}(\Omega)+\left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}(\Omega)\boldsymbol{\mathcal{E}}+\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}(\Omega)(\boldsymbol{\Theta}-\bar{\boldsymbol{\Theta}})+\bar{\boldsymbol{X}}_{\Lambda}(\Omega)(\Lambda-\bar{\Lambda}(\Omega))+\bar{\boldsymbol{X}}_{\sigma}\right)\sigma+\mathcal{O}(\sigma^2),$$

while second-order expansions can be defined analogously.

B.3 Proof of Lemma 1 and Lemma 2

We begin by showing that when $\sigma=0$, $\tilde{\boldsymbol{z}}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=\boldsymbol{z}$ and hence that $\tilde{\Omega}(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0)=\Omega$. From (47) it is clear that $\tilde{r}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=\frac{\mu}{\beta}$ and thus equation (48) implies

$$\tilde{\mu}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=\mu.$$

In absence of shocks, we conclude from equations (41) and (49) that

$$\tilde{c}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=\tilde{m}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)^{1/\nu}\tilde{C}(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0)$$

and

$$\tilde{c}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=m^{1/\nu}\left(\frac{\beta}{\tilde{\mathcal{Q}}(\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0)(1+\Pi(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0))}\right)^{1/\nu}$$

As $\int m^{\frac{1}{\nu}}d\Omega(\boldsymbol{z})=1$, this implies $\left(\frac{\beta}{\tilde{\mathcal{Q}}(\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0)(1+\Pi(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0))}\right)^{1/\nu}=\tilde{C}(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0)$ and hence

$$\tilde{c}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=m^{\frac{1}{\nu}}\tilde{C}(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0).$$

We, therefore, conclude that $\tilde{m}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=m$.

By assumption $\tilde{s}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=s$ and, as $\rho_\theta\to1$ as $\sigma\to0$, $\tilde{\theta}(0,0,0,\boldsymbol{z},\Omega,\bar{\Lambda};0)=\theta$. All combined, we obtain $\tilde{\boldsymbol{z}}(0,0,\bar{\boldsymbol{\Theta}},\boldsymbol{z},\Omega,\bar{\Lambda};0)=\boldsymbol{z}$ and therefore $\tilde{\Omega}(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0)=\Omega$. Note that we obtained $\tilde{\Omega}(0,\bar{\boldsymbol{\Theta}},\Omega,\bar{\Lambda};0)=\Omega$ by exploiting that, in absence of risk, the expectations in (41) and (49) drop out. This would hold apply along any deterministic transition path for $(\Lambda,\boldsymbol{\Theta})$ and thus $\tilde{\boldsymbol{z}}(0,0,\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda;0)=\boldsymbol{z}$ and $\tilde{\Omega}(0,\boldsymbol{\Theta},\Omega,\Lambda;0)=\Omega$ for any $(\Lambda,\boldsymbol{\Theta})$. This proves the statement of lemma 2 and, therefore, also of lemma 1 as it is a special case of lemma 2.

B.4 Frechet Derivatives: $\partial\bar{\boldsymbol{x}}(\boldsymbol{z})$, $\partial\bar{\boldsymbol{X}}$

Differentiate F and R with respect to $\boldsymbol{z}$ to obtain

$$(\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z})+\mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})+\mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z}))\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})+\mathsf{F}_{\boldsymbol{z}}(\boldsymbol{z})=0,$$

where we exploited $\bar{\boldsymbol{z}}_{\boldsymbol{z}} = I$. This yields $\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}) = (\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z}))^{-1}\mathsf{F}_{\boldsymbol{z}}(\boldsymbol{z})$.

We begin by solving for the derivatives w.r.t Λ. Exploiting $\tilde{\boldsymbol{z}}(0,0,0,\boldsymbol{z},\Omega,\Lambda;0) = \boldsymbol{z}$ and $\tilde{\Omega}(0,0,\Omega,\Lambda;0) = \Omega$ and hence $\bar{\boldsymbol{z}}_\Lambda = 0$ and $\bar{\Omega}_\Lambda = 0$,

$$\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\bar{\Lambda}_\Lambda\right) + \mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda = 0$$

and

$$\int \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda + \mathsf{R}_\Lambda(\boldsymbol{z})d\Omega(\boldsymbol{z}) = 0.$$

In order to proceed, we need to find the derivative $\bar{\Lambda}_\Lambda$. This requires solving a nonlinear equation but still involves only operations involving small matrices. First note that

$$\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) = -(\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \bar{\Lambda}_\Lambda\mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z}))^{-1}\mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda$$

Let $\mathsf{C}(\boldsymbol{z}) = -(\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \bar{\Lambda}_\Lambda\mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z}))^{-1}\mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})$, then

$$\bar{\boldsymbol{X}}_\Lambda = -\left(\int \left[\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\mathsf{C}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z}) + \bar{\Lambda}_\Lambda\mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\right]d\Omega(\boldsymbol{z})\right)^{-1}\left(\int \mathsf{R}_\Lambda(\boldsymbol{z})d\Omega(\boldsymbol{z})\right).$$

Therefore, $\bar{\Lambda}_\Lambda$ must solve

$$\bar{\Lambda}_\Lambda = -\mathsf{P}\left(\int \left[\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\mathsf{C}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z}) + \bar{\Lambda}_\Lambda\mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\right]d\Omega(\boldsymbol{z})\right)^{-1}\left(\int \mathsf{R}_\Lambda(\boldsymbol{z})d\Omega(\boldsymbol{z})\right).$$

This can be found easily with a 1-dimensional root solver as all the inversions involve small matrices.

The Frechet derivative w.r.t Ω is computed as follows:

$$(\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z}))\,\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\mathsf{P}\partial\bar{\boldsymbol{X}} + \mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}} = 0$$

which gives

$$\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) = -(\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z}))^{-1}\left(\mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\mathsf{P} + \mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\right)\partial\bar{\boldsymbol{X}} \equiv \mathsf{C}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}. \quad (74)$$

To solve for $\partial\bar{\boldsymbol{X}}$, we differentiate R in the direction Δ, with density δ, to obtain

$$\begin{aligned}0 &= \lim_{\alpha\to 0}\frac{1}{\alpha}\left[\int R(\boldsymbol{z},\bar{\boldsymbol{x}}(\boldsymbol{z},\Omega+\alpha\Delta),\bar{\boldsymbol{X}}(\Omega+\alpha\Delta))(\omega(\boldsymbol{z})+\alpha\delta(\boldsymbol{z}))d\boldsymbol{z} - \int \mathsf{R}(\boldsymbol{z})\omega(\boldsymbol{z})d\boldsymbol{z}\right]\\ &= \int \mathsf{R}(\boldsymbol{z})\delta(\boldsymbol{z})d\boldsymbol{z} + \int \left(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left((\partial\bar{\boldsymbol{X}}\cdot\boldsymbol{\Delta}) + \bar{\boldsymbol{X}}_\Lambda\mathsf{P}(\partial\bar{\boldsymbol{X}}\cdot\boldsymbol{\Delta})\right) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\Delta\right)\omega(\boldsymbol{z})d\boldsymbol{z}\end{aligned}$$

where $\mathsf{R}(\boldsymbol{z})$ is defined by $\mathsf{R}(\boldsymbol{z}) = R(\boldsymbol{z}, \bar{\boldsymbol{x}}(z), \bar{\boldsymbol{X}}, \bar{\boldsymbol{X}}, 0, \bar{\Lambda}, \bar{\boldsymbol{\Theta}})$. Substituting for $\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) = \mathsf{C}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}$, we get

$$\partial\bar{\boldsymbol{X}} \cdot \Delta = -\left(\int \left(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\mathsf{C}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})(I + \bar{\boldsymbol{X}}_\Lambda\mathsf{P}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\right) d\Omega(\boldsymbol{z})\right)^{-1} \int \mathsf{R}(\boldsymbol{z})d\Delta(\boldsymbol{z})$$
$$\equiv \mathsf{D}^{-1} \int \mathsf{R}(\boldsymbol{z})d\Delta(\boldsymbol{z}). \tag{75}$$

B.5 First-Order Terms

Next we differentiate both F and R with respect to σ and use the method of undetermined coefficients to group the terms that multiply $\boldsymbol{\varepsilon}$, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Theta}$ and set each equal to zero. For $\boldsymbol{\varepsilon}$, this yields

$$\mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}) = 0$$

or

$$\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}) = -\left(\mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\right)^{-1}\mathsf{F}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}) \equiv \mathsf{E}(\boldsymbol{z})^{-1}\mathsf{G}(\boldsymbol{z}). \tag{76}$$

For $\boldsymbol{\Theta}$, we find that (after noting imposing $\bar{\Omega}_{\boldsymbol{\Theta}} = 0$ and $\bar{\boldsymbol{z}}_{\boldsymbol{\Theta}} = 0$)

$$\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z})\rho_{\boldsymbol{\Theta}} + \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\mathsf{P}\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}\right) + \mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} + \mathsf{F}_{\boldsymbol{\Theta}}(\boldsymbol{z}) = 0.$$

This yields a linear equation in $\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}$ and $\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}$ which we can solve for $\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}$.[20] Plugging in for the linear relationship between $\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}$ and $\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}$ in

$$\int \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}\rho_{\boldsymbol{\Theta}} + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda\mathsf{P}\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} + \mathsf{R}_{\boldsymbol{\Theta}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) = 0.$$

yields a linear equation for $\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}$.

Finally, for $\boldsymbol{\mathcal{E}}$ we find

$$\mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z}) + \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) + \partial\bar{\boldsymbol{x}}(\boldsymbol{z}) \cdot \bar{\Omega}_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\mathsf{P}\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}\right) + \mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} + \mathsf{F}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) = 0$$

and

$$\int \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left(\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} + \partial\bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{X}}_\Lambda\mathsf{P}\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}\right) + \mathsf{R}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) = 0.$$

[20] Easiest to exploit $\rho_{\boldsymbol{\Theta}} = \begin{pmatrix} \rho_\Theta & 0 \\ 0 & \rho_\Phi \end{pmatrix}$ and solve for each column of $\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}$ separately.

Substituting for $\partial\bar{\boldsymbol{x}}$, we obtain and defining $\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}} = \partial\bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\boldsymbol{\mathcal{E}}}$

$$\mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})+\mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z}) + \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) + \mathsf{C}(\boldsymbol{z})\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{x}}_{\Lambda}\mathsf{P}\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}\right)+\mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}+\mathsf{F}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) = 0,$$

or

$$\mathsf{H}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) = \mathsf{I}(\boldsymbol{z})\begin{bmatrix} I & \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} & \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}} \end{bmatrix}^{\mathsf{T}} \tag{77}$$

Solving for these derivatives first requires determining $\bar{\Omega}_{\boldsymbol{\mathcal{E}}}$. Differentiating equation (73) with respect to $\boldsymbol{\mathcal{E}}$ and letting $\boldsymbol{z}^i$ represent the i^{th} element of $\boldsymbol{z}$ yields

$$\begin{aligned}\tilde{\Omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \Omega, \Lambda)(\boldsymbol{y}) = -\int \sum_i \delta(\tilde{\boldsymbol{z}}^i(\varepsilon, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) - \boldsymbol{y}^i) \prod_{j\neq i} \iota\left(\tilde{\boldsymbol{z}}^j(\varepsilon, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) \leq \boldsymbol{y}^j\right) \\ \times \tilde{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}(\varepsilon, \boldsymbol{\mathcal{E}}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega)\, d\Pr(\varepsilon) d\Omega(\boldsymbol{z})\end{aligned}$$

which evaluated at $\sigma = 0$ gives

$$\bar{\Omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y}) = -\int \sum_i \delta(\boldsymbol{z}^i - \boldsymbol{y}^i) \prod_{j\neq i} \iota(\boldsymbol{z}^j \leq \boldsymbol{y}^j)\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) = -\sum_i \int \delta(\boldsymbol{z}^i - \boldsymbol{y}^i) \prod_{j\neq i} \iota(\boldsymbol{z}^j \leq \boldsymbol{y}^j)\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}).$$

The density of $\bar{\Omega}_{\boldsymbol{\mathcal{E}}}$ is given by $\bar{\omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y}) = \frac{\partial^{n\boldsymbol{z}}}{\partial \boldsymbol{y}^1 \partial \boldsymbol{y}^2 \cdots \partial \boldsymbol{y}^{n\boldsymbol{z}}}\bar{\Omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y})$ so

$$\bar{\omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y}) = -\sum_i \frac{\partial}{\partial \boldsymbol{y}^i} \int \prod_j \delta(\boldsymbol{z}^j - \boldsymbol{y}^j)\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) = -\sum_i \frac{\partial}{\partial \boldsymbol{y}^i}\left(\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y})\omega(\boldsymbol{y})\right).$$

Plugging in for the definition of $\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}}$ we find

$$\begin{aligned}\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}} &= -\mathsf{D}^{-1}\int \mathsf{R}(\boldsymbol{z}) \sum_i \frac{\partial}{\partial \boldsymbol{z}^i}\left(\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\omega(\boldsymbol{z})\right) d\boldsymbol{z} \\ &= \mathsf{D}^{-1}\int \sum_i \left(\frac{\partial}{\partial \boldsymbol{z}^i}\mathsf{R}(\boldsymbol{z})\right)\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\omega(\boldsymbol{z})d\boldsymbol{z} \\ &= \mathsf{D}^{-1}\int \sum_i \left(\mathsf{R}_{\boldsymbol{z}^i}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{z}^i}(\boldsymbol{z})\right)\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}d\Omega(\boldsymbol{z}) \\ &= \mathsf{D}^{-1}\int \left(\mathsf{R}_{\boldsymbol{z}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\right)\mathsf{p}\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) \\ &\equiv \int \mathsf{A}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z})\end{aligned} \tag{78}$$

where we arrive at the second equality through integration by parts. This implies

$$\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) \cdot \bar{\Omega}_{\boldsymbol{\mathcal{E}}} = \mathsf{C}(\boldsymbol{z})\int \mathsf{A}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}). \tag{79}$$

Substituting for $\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})$ using equation (77) in (78) plus

$$\int \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} + \mathsf{R}_{\boldsymbol{X}+}\left(\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} + \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{X}}_{\Lambda}\mathsf{P}\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}\right) + \mathsf{R}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) = 0$$

yields a linear system

$$\mathsf{K}\cdot\left[\begin{array}{cc}\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} & \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}}\end{array}\right]^{\mathsf{T}} = \mathsf{J} \tag{80}$$

where K and J are matrices with dimensions given by the number of aggregate variables or number of aggregate shocks, both of which are small relative to number of agents.

Finally our last term, the derivative w.r.t σ. Since we have the AR(1) parameter of idiosyncratic shock ρ_θ scale with σ, F_σ is non-zero.

$$\begin{aligned}&\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z})\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\bar{\Omega}_\sigma + \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\mathsf{P}\bar{\boldsymbol{X}}_\sigma\right)\\&+\mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\sigma + \mathsf{F}_\sigma(\boldsymbol{z}) = 0\end{aligned}$$

and[21]

$$\int \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\sigma + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left(\bar{\boldsymbol{X}}_\sigma + \partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_\sigma + \bar{\boldsymbol{X}}_\Lambda\mathsf{P}\bar{\boldsymbol{X}}_\sigma\right)d\Omega(\boldsymbol{z}) = 0.$$

Substituting for $\partial\bar{\boldsymbol{x}}$, we obtain and defining $\bar{\boldsymbol{X}}'_\sigma = \partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_\sigma$

$$\begin{aligned}&\mathsf{F}_{\boldsymbol{x}-}(\boldsymbol{z})\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \mathsf{F}_{\boldsymbol{x}+}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \mathsf{C}(\boldsymbol{z})\bar{\boldsymbol{X}}'_\sigma + \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\mathsf{P}\bar{\boldsymbol{X}}_\sigma\right)\\&+\mathsf{F}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\sigma + \mathsf{F}_\sigma(\boldsymbol{z}) = 0,\end{aligned}$$

or

$$\mathsf{L}(\boldsymbol{z})\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) = \mathsf{M}(\boldsymbol{z})\left[\begin{array}{ccc}I & \bar{\boldsymbol{X}}_\sigma & \bar{\boldsymbol{X}}'_\sigma\end{array}\right]^{\mathsf{T}}. \tag{81}$$

Differentiating with respect to σ yields

$$\begin{aligned}\tilde{\Omega}_\sigma(\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\Omega,\Lambda)(\boldsymbol{y}) = -\int\sum_i \delta(\tilde{\boldsymbol{z}}^i(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda) - \boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{\boldsymbol{z}}^j(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\leq\boldsymbol{y}^j\right)\\ \times\left(\tilde{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega)\,\varepsilon + \tilde{\boldsymbol{z}}^i_\sigma(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega)\right)d\Pr(\varepsilon)d\Omega(\boldsymbol{z})\end{aligned}$$

which evaluated at $\sigma = 0$ gives (as $\boldsymbol{\varepsilon}$ is mean 0)

$$\bar{\Omega}_\sigma(\boldsymbol{y}) = -\sum_i\int\delta(\boldsymbol{z}^i - \boldsymbol{y}^i)\prod_{j\neq i}\iota(\boldsymbol{z}^j\leq\boldsymbol{y}^j)\bar{\boldsymbol{z}}^i_{\boldsymbol{\sigma}}(\boldsymbol{z})d\Omega(\boldsymbol{z}).$$

[21]Note, we have dropped all the expectations as they are zero.

and

$$\bar{\omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y}) = -\sum_i \frac{\partial}{\partial \boldsymbol{y}^i} \left(\bar{z}_\sigma^i(\boldsymbol{y})\omega(\boldsymbol{y})\right).$$

Plugging in for the definition of $\bar{\boldsymbol{X}}'_\sigma$ we find

$$\bar{\boldsymbol{X}}'_\sigma = \mathsf{D}^{-1} \int \left(\mathsf{R}_{\boldsymbol{z}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\right) \mathsf{p}\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) d\Omega(\boldsymbol{z}) \tag{82}$$

Substituting for $\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z})$ in (82) and

$$\int \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\sigma(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\sigma + \mathsf{R}_{\boldsymbol{X}+}\left(\bar{\boldsymbol{X}}_\sigma + \bar{\boldsymbol{X}}'_\sigma + \bar{\boldsymbol{X}}_\Lambda \mathsf{P}\bar{\boldsymbol{X}}_\sigma\right) + \mathsf{R}_\sigma(\boldsymbol{z}) d\Omega(\boldsymbol{z}) = 0$$

yields a linear system

$$\mathsf{N} \cdot \left[\begin{array}{cc} \bar{\boldsymbol{X}}_\sigma & \bar{\boldsymbol{X}}'_\sigma \end{array}\right]^{\mathsf{T}} = \mathsf{O} \tag{83}$$

which is composed of small matrices.

B.6 Second-Order Terms

Here we describe how to compute all the second-order terms required for the Taylor expansion. To save on space, when obvious we drop dependence on $\boldsymbol{z}$.

B.6.1 Derivatives w.r.t. states

We start by differentiating with respect to the states $\boldsymbol{z}$ and Ω. The term $\partial\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})$ can be computed by differentiating F w.r.t. $\boldsymbol{z}$ and then taking the Frechet derivative with respect to Ω to find

$$\begin{aligned}
0 = & \mathsf{F}_{\boldsymbol{x}-}\partial\bar{\boldsymbol{x}}_{\boldsymbol{z}} + \mathsf{F}_{\boldsymbol{x}}\partial\bar{\boldsymbol{x}}_{\boldsymbol{z}} + \mathsf{F}_{\boldsymbol{x}+}\left(\partial\bar{\boldsymbol{x}}_{\boldsymbol{z}} + \bar{\boldsymbol{x}}_{\boldsymbol{z}\Lambda}\partial\bar{\Lambda}\right) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{X}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{X}}\right) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{X}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{X}}\right) \\
& + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{X}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{X}}\right) + \mathsf{F}_{\boldsymbol{z}\boldsymbol{x}-} \cdot \left(I, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{z}\boldsymbol{x}} \cdot \left(I, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{z}\boldsymbol{X}} \cdot \left(I, \partial\bar{\boldsymbol{X}}\right) + \mathsf{F}_{\boldsymbol{z}\boldsymbol{x}+} \cdot \left(I, \partial\bar{\boldsymbol{x}}^+\right) \\
& + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}-} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{X}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{X}}\right) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}+} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}^+\right) \\
& + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}-} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{X}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{X}}\right) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}+} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}^+\right) \\
& + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}-} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}\right) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{X}} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{X}}\right) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}+} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{z}}, \partial\bar{\boldsymbol{x}}^+\right)
\end{aligned}$$

where I represents the identity matrix and we use $\mathsf{a} \cdot (\mathsf{b}, \mathsf{c})$ to denote a bilinear map[22] and $\partial\bar{\boldsymbol{x}}^+(\boldsymbol{z}) \equiv \partial\bar{\boldsymbol{x}}(\boldsymbol{z}) + \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\partial\bar{\Lambda}$. This linear system is easily solved to give

$$\partial\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}) = \mathsf{Q}(\boldsymbol{z}) \cdot \left(I, \partial\bar{\boldsymbol{X}}\right) \tag{84}$$

[22]Specifically, if a is a $n_1 \times n_2 \times n_3$ tensor, b is a $n_2 \times n_4$ matrix and c is a $n_3 \times n_5$ matrix

where $\mathsf{Q}(\boldsymbol{z})$ is $n_{\boldsymbol{x}} \times n_{\boldsymbol{z}} \times n_{\boldsymbol{X}}$. The derivatives of $\bar{\boldsymbol{x}}_{\boldsymbol{zz}}(\boldsymbol{z})$, $\bar{\boldsymbol{x}}_{\boldsymbol{z}\Lambda}(\boldsymbol{z})$ and $\bar{\boldsymbol{x}}_{\boldsymbol{z\Theta}}(\boldsymbol{z})$ are equally easy as they do not effect aggregates.

Next, twice differentiating with respect to Λ (again with the knowledge that $\bar{\boldsymbol{z}}_{\Lambda\Lambda} = 0$ and $\bar{\Omega}_{\Lambda\Lambda} = 0$) implies

$$\begin{aligned}0 =&\mathsf{F}_{\boldsymbol{x}-}\bar{\boldsymbol{x}}_{\Lambda\Lambda} + \mathsf{F}_{\boldsymbol{x}}\bar{\boldsymbol{x}}_{\Lambda\Lambda} + \mathsf{F}_{\boldsymbol{X}}\bar{\boldsymbol{X}}_{\Lambda\Lambda} + \mathsf{F}_{\boldsymbol{x}+}\left(\bar{\boldsymbol{x}}_{\Lambda}\mathsf{P}\bar{\boldsymbol{X}}_{\Lambda\Lambda} + \bar{\boldsymbol{x}}_{\Lambda\Lambda}\bar{\Lambda}_{\Lambda}^2\right)\\ &+ \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}-}\cdot(\bar{\boldsymbol{x}}_{\Lambda},\bar{\boldsymbol{x}}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}}\cdot(\bar{\boldsymbol{x}}_{\Lambda},\bar{\boldsymbol{x}}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{X}}(\bar{\boldsymbol{x}}_{\Lambda},\bar{\boldsymbol{X}}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}+}\cdot(\bar{\boldsymbol{x}}_{\Lambda},\bar{\boldsymbol{x}}_{\Lambda}\bar{\Lambda}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{x}-\Lambda}\bar{\boldsymbol{x}}_{\Lambda}\\ &+ \mathsf{F}_{\boldsymbol{xx}}\cdot(\bar{\boldsymbol{x}}_{\Lambda},\bar{\boldsymbol{x}}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{xX}}\cdot(\bar{\boldsymbol{x}}_{\Lambda},\bar{\boldsymbol{X}}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{xx}+}\cdot(\bar{\boldsymbol{x}}_{\Lambda},\bar{\boldsymbol{x}}_{\Lambda}\bar{\Lambda}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{x}\Lambda}\bar{\boldsymbol{x}}_{\Lambda} + \mathsf{F}_{\boldsymbol{XX}}\cdot(\bar{\boldsymbol{X}}_{\Lambda},\bar{\boldsymbol{X}}_{\Lambda})\\ &+ 2\mathsf{F}_{\boldsymbol{Xx}+}\cdot(\bar{\boldsymbol{X}}_{\Lambda},\bar{\boldsymbol{x}}_{\Lambda}\bar{\Lambda}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{X}\Lambda}\bar{\boldsymbol{X}}_{\Lambda} + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}+}\cdot(\bar{\boldsymbol{x}}_{\Lambda}\bar{\Lambda}_{\Lambda},\bar{\boldsymbol{x}}_{\Lambda}\bar{\Lambda}_{\Lambda}) + 2\mathsf{F}_{\boldsymbol{x}+\Lambda}\bar{\boldsymbol{x}}_{\Lambda}\bar{\Lambda}_{\Lambda} + \mathsf{F}_{\Lambda\Lambda}\end{aligned}$$

and

$$\begin{aligned}0 = \int\Bigg(&\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\Lambda\Lambda}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\Lambda\Lambda} + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left(\bar{\boldsymbol{X}}_{\Lambda}\mathsf{P}\bar{\boldsymbol{X}}_{\Lambda\Lambda} + \bar{\boldsymbol{X}}_{\Lambda\Lambda}\bar{\Lambda}_{\Lambda}^2\right) + \mathsf{R}_{\boldsymbol{xx}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z}),\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z}))\\ &+ 2\mathsf{R}_{\boldsymbol{xX}}(\boldsymbol{z})\cdot\left(\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z}),\bar{\boldsymbol{X}}_{\Lambda}\right) + 2\mathsf{R}_{\boldsymbol{xX}+}(\boldsymbol{z})\cdot\left(\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z}),\bar{\boldsymbol{X}}_{\Lambda}\bar{\Lambda}_{\Lambda}\right) + 2\mathsf{R}_{\boldsymbol{x}\Lambda}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z})\\ &+ \mathsf{R}_{\boldsymbol{XX}}(\boldsymbol{z})\cdot\left(\bar{\boldsymbol{X}}_{\Lambda},\bar{\boldsymbol{X}}_{\Lambda}\right) + 2\mathsf{R}_{\boldsymbol{XX}+}(\boldsymbol{z})\cdot\left(\bar{\boldsymbol{X}}_{\Lambda},\bar{\boldsymbol{X}}_{\Lambda}\bar{\Lambda}_{\Lambda}\right) + 2\mathsf{R}_{\boldsymbol{X}\Lambda}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\Lambda}\\ &+ \mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}+}\cdot\left(\bar{\boldsymbol{X}}_{\Lambda}\bar{\Lambda}_{\Lambda},\bar{\boldsymbol{X}}_{\Lambda}\bar{\Lambda}_{\Lambda}\right) + 2\mathsf{R}_{\boldsymbol{X}+\Lambda}\bar{\boldsymbol{X}}_{\Lambda}\bar{\Lambda}_{\Lambda} + \mathsf{R}_{\Lambda\Lambda}(\boldsymbol{z})\Bigg)d\Omega(\boldsymbol{z})\end{aligned}$$

From the first set of equations, we obtain

$$\mathsf{S}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\Lambda\Lambda}(\boldsymbol{z}) = \mathsf{T}(\boldsymbol{z})\begin{bmatrix} I & \bar{\boldsymbol{X}}_{\Lambda\Lambda}\end{bmatrix} \tag{85}$$

which we can then plug into the R equation to yield

$$\mathsf{U}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\Lambda\Lambda} = \mathsf{V}(\boldsymbol{z}). \tag{86}$$

A identical approach can be used to obtain expressions for $\bar{\boldsymbol{x}}_{\boldsymbol{\Theta\Theta}}(\boldsymbol{z}), \bar{\boldsymbol{x}}_{\boldsymbol{\Theta}\Lambda}(\boldsymbol{z}), \bar{\boldsymbol{X}}_{\boldsymbol{\Theta\Theta}}$ and $\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}\Lambda}$ which take the same form.

Next $\partial\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z})$ is computed by differentiating F and R w.r.t Λ and then taking the Frechet

---

then $\mathsf{d} = \mathsf{a}\cdot(\mathsf{b},\mathsf{c})$ is $n_1 \times n_4 \times n_5$ tensor defined by

$$\mathsf{d}_{ilm} = \sum_{j,k}\mathsf{a}_{ijk}\mathsf{b}_{jl}\mathsf{c}_{km}.$$

This definition generalizes to when a,b, or c is infinite dimensional, such as with $\partial\bar{\boldsymbol{x}}_{\mathsf{z}}$.

derivative with respect to Ω to find (exploiting $\partial\bar{\Omega}_\Lambda = 0$)

$$
\begin{aligned}
0 =& \mathsf{F}_{\boldsymbol{x}-}\partial\bar{\boldsymbol{x}}_\Lambda + \mathsf{F}_{\boldsymbol{x}}\partial\bar{\boldsymbol{x}}_\Lambda + \mathsf{F}_{\boldsymbol{X}}\partial\bar{\boldsymbol{X}}_\Lambda + \mathsf{F}_{\boldsymbol{x}+}\left(\bar{\boldsymbol{x}}_\Lambda \mathsf{P}\partial\bar{\boldsymbol{X}}_\Lambda + \partial\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda + \bar{\boldsymbol{x}}_{\Lambda\Lambda}\cdot\left(\bar{\Lambda}_\Lambda, \partial\bar{\Lambda}\right)\right) \\
&+ \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}-}\cdot(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{x}}) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}}\cdot(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{x}}) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{X}}(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{X}}) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}+}\cdot(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{x}}^+) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}-}\cdot(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{x}}) \\
&+ \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}}\cdot(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{x}}) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{X}}\cdot(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{X}}) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}+}\cdot(\bar{\boldsymbol{x}}_\Lambda, \partial\bar{\boldsymbol{x}}^+) + \mathsf{F}_{\boldsymbol{X}\boldsymbol{x}-}\cdot(\bar{\boldsymbol{X}}_\Lambda, \partial\bar{\boldsymbol{x}}) + \mathsf{F}_{\boldsymbol{X}\boldsymbol{x}}\cdot(\bar{\boldsymbol{X}}_\Lambda, \partial\bar{\boldsymbol{x}}) \\
&+ \mathsf{F}_{\boldsymbol{X}\boldsymbol{x}+}\cdot(\bar{\boldsymbol{X}}_\Lambda, \partial\bar{\boldsymbol{x}}^+) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}-}\cdot(\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda, \partial\bar{\boldsymbol{x}}) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}}\cdot(\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda, \partial\bar{\boldsymbol{x}}) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{X}}\cdot(\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda, \partial\bar{\boldsymbol{X}}) \\
&+ \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}+}\cdot(\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda, \partial\bar{\boldsymbol{x}}^+)
\end{aligned}
$$

and

$$
\begin{aligned}
0 =& \int\left(\mathsf{R}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda\right)\delta(\boldsymbol{z})d\boldsymbol{z} \\
&+ \int\Bigg(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\cdot\Delta + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left(\bar{\boldsymbol{X}}_\Lambda\mathsf{P}\partial\bar{\boldsymbol{X}}_\Lambda\cdot\Delta + \partial\bar{\boldsymbol{X}}_\Lambda\cdot\Delta\bar{\Lambda}_\Lambda + \bar{\boldsymbol{X}}_{\Lambda\Lambda}\cdot\left(\bar{\Lambda}_\Lambda, \partial\bar{\Lambda}\cdot\Delta\right)\right) \\
&+ \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}_\Lambda\cdot\Delta + \mathsf{R}_{\boldsymbol{x}\boldsymbol{x}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}), \partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}), \partial\bar{\boldsymbol{X}}\cdot\Delta) \\
&+ \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}+}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}), \partial\bar{\boldsymbol{X}}^+\cdot\Delta) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{x}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{X}}_\Lambda, \partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{X}}_\Lambda, \partial\bar{\boldsymbol{X}}\cdot\Delta) \\
&+ \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}+}(\boldsymbol{z})\cdot(\bar{\boldsymbol{X}}_\Lambda, \partial\bar{\boldsymbol{X}}^+\cdot\Delta) + \mathsf{R}_{\boldsymbol{X}+\boldsymbol{x}}(\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda, \partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta) + \mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}}\cdot(\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda, \partial\bar{\boldsymbol{X}}\cdot\Delta) \\
&+ \mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}+}\cdot(\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda, \partial\bar{\boldsymbol{X}}^+\cdot\Delta) + \mathsf{R}_{\Lambda\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta + \mathsf{R}_{\Lambda\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\Delta + \mathsf{R}_{\Lambda\boldsymbol{X}+}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^+\cdot\Delta\Bigg)d\Omega(\boldsymbol{z}),
\end{aligned}
$$

where $\partial\bar{\boldsymbol{X}}^+ \equiv \partial\bar{\boldsymbol{X}} + \bar{\boldsymbol{X}}_\Lambda\partial\bar{\Lambda}$. From F equations we immediately obtain

$$
\mathsf{W}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) = \mathsf{X}(\boldsymbol{z})\left[\begin{array}{cc}\partial\bar{\boldsymbol{X}}_\Lambda & \partial\bar{\boldsymbol{X}}\end{array}\right]^{\mathsf{T}} \tag{87}
$$

which can be plugged into the equations from R to obtain

$$
\mathsf{Y}\partial\bar{\boldsymbol{X}}_\Lambda\cdot\Delta = \mathsf{Z}\partial\bar{\boldsymbol{X}}\cdot\Delta + \int\left(\mathsf{R}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda\right)\delta(\boldsymbol{z})d\boldsymbol{z}. \tag{88}
$$

A similar approach can be used to obtain expressions for $\partial\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z})$ and $\partial\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}$ which take the same form.

Finally the second-order Frechet derivative of F in the directions of Δ_1 and Δ_2 yields

$$
\begin{aligned}
0 = & \mathsf{F}_{\boldsymbol{x}-}\partial^2\bar{\boldsymbol{x}}\cdot(\Delta_1,\Delta_2) + \mathsf{F}_{\boldsymbol{x}}\partial^2\bar{\boldsymbol{x}}\cdot(\Delta_1,\Delta_2) + \mathsf{F}_{\boldsymbol{x}+}\partial^2\bar{\boldsymbol{x}}\cdot(\Delta_1,\Delta_2) + \mathsf{F}_{\boldsymbol{X}}\partial^2\bar{\boldsymbol{X}}\cdot(\Delta_1,\Delta_2) \\
& + \mathsf{F}_{\boldsymbol{x}+}\left(\bar{\boldsymbol{x}}_{\Lambda}\mathsf{P}\partial^2\bar{\boldsymbol{X}}\cdot(\Delta_1,\Delta_2) + \bar{\boldsymbol{x}}_{\Lambda\Lambda}\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) + (\partial\bar{\boldsymbol{x}}_{\Lambda}\cdot\Delta_1)\left(\partial\bar{\Lambda}\cdot\Delta_2\right) + (\partial\bar{\boldsymbol{x}}_{\Lambda}\cdot\Delta_2)\left(\partial\bar{\Lambda}\cdot\Delta_1\right)\right) \\
& + \mathsf{F}_{\mathbf{x}-\mathbf{x}-}\cdot(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2) + \mathsf{F}_{\mathbf{x}-\mathbf{x}}\cdot(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2) + \mathsf{F}_{\mathbf{x}-\mathbf{x}+}\cdot\left(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}^+\cdot\Delta_2\right) \\
& + \mathsf{F}_{\mathbf{x}-\mathbf{X}}\cdot\left(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) + \mathsf{F}_{\mathbf{x}\mathbf{x}-}\cdot(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2) + \mathsf{F}_{\mathbf{x}\mathbf{x}}\cdot(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2) \\
& + \mathsf{F}_{\mathbf{x}\mathbf{x}+}\cdot\left(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}^+\cdot\Delta_2\right) + \mathsf{F}_{\mathbf{x}\mathbf{X}}\cdot\left(\partial\bar{\boldsymbol{x}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) + \mathsf{F}_{\mathbf{x}+\mathbf{x}-}\cdot\left(\partial\bar{\boldsymbol{x}}^+\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2\right) \\
& + \mathsf{F}_{\mathbf{x}+\mathbf{x}}\cdot\left(\partial\bar{\boldsymbol{x}}^+\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2\right) + \mathsf{F}_{\mathbf{x}+\mathbf{x}+}\cdot\left(\partial\bar{\boldsymbol{x}}^+\cdot\Delta_1,\partial\bar{\boldsymbol{x}}^+\cdot\Delta_2\right) + \mathsf{F}_{\mathbf{x}+\mathbf{X}}\cdot\left(\partial\bar{\boldsymbol{x}}^+\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) \\
& + \mathsf{F}_{\boldsymbol{X}\mathbf{x}-}\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2\right) + \mathsf{F}_{\mathbf{X}\mathbf{x}}\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}\cdot\Delta_2\right) + \mathsf{F}_{\mathbf{X}\mathbf{x}+}\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}^+\cdot\Delta_2\right) \\
& + \mathsf{F}_{\mathbf{X}\mathbf{X}}\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right)
\end{aligned}
$$

Substituting for $\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) = \mathsf{C}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}$ and $\partial\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z})$, and then solving for $\partial^2\bar{\boldsymbol{x}}$ gives

$$
\begin{aligned}
\partial^2\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot(\Delta_1,\Delta_2) = & \mathsf{C}(\boldsymbol{z})\partial^2\bar{\boldsymbol{X}}\cdot(\Delta_1,\Delta_2) + \mathsf{AA}_1(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) \\
& + \mathsf{AA}_2(\boldsymbol{z})\left(\partial\bar{\boldsymbol{X}}_{\Lambda}\cdot\Delta_1\partial\bar{\Lambda}\cdot\Delta_2 + \partial\bar{\boldsymbol{X}}_{\Lambda}\cdot\Delta_2\partial\bar{\Lambda}\cdot\Delta_1\right).
\end{aligned} \tag{89}
$$

To find $\partial^2\bar{\boldsymbol{X}}$ we differentiate R w.r.t Ω in the direction Δ_1 and then Δ_2 to find

$$
\begin{aligned}
0 = & \int\Bigg(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\partial^2\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot(\Delta_1,\Delta_2) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\partial^2\bar{\boldsymbol{X}}\cdot(\Delta_1,\Delta_2) + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\partial^2\bar{\boldsymbol{X}}\cdot(\Delta_1,\Delta_2) \\
& + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left(\bar{\boldsymbol{X}}_{\Lambda\Lambda}(\partial\bar{\Lambda}\cdot\Delta_1)(\partial\bar{\Lambda}\cdot\Delta_2) + \left(\partial\bar{\boldsymbol{X}}_{\Lambda}\cdot\Delta_1\right)\partial\bar{\Lambda}\cdot\Delta_2 + \left(\partial\bar{\boldsymbol{X}}_{\Lambda}\cdot\Delta_2\right)\partial\bar{\Lambda}\cdot\Delta_1\right) \\
& + \mathsf{R}_{\boldsymbol{x}\boldsymbol{x}}(\boldsymbol{z})\cdot(\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_1,\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_2) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) \\
& + \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}+}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_1,\partial\bar{\boldsymbol{X}}^+\cdot\Delta_2\right) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{x}}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_2\right) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) \\
& + \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}+}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}^+\cdot\Delta_2\right) + \mathsf{R}_{\boldsymbol{X}+\boldsymbol{x}}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{X}}^+\cdot\Delta_1,\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_2\right) \\
& + \mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{X}}^+\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) + \mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}+}(\boldsymbol{z})\cdot\left(\partial\bar{\boldsymbol{X}}^+\cdot\Delta_1,\partial\bar{\boldsymbol{X}}^+\cdot\Delta_2\right)\Bigg)d\Omega(\boldsymbol{z}) \\
& + \int\left(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_1 + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\Delta_1 + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^+\cdot\Delta_1\right)d\Delta_2(\boldsymbol{z}) \\
& + \int\left(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_2 + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\Delta_2 + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^+\cdot\Delta_2\right)d\Delta_1(\boldsymbol{z})
\end{aligned}
$$

A bit of rearranging and substituting for $\partial^2\bar{\boldsymbol{x}}$ yields

$$
\begin{aligned}
\mathsf{D}\partial^2\bar{\boldsymbol{X}}\cdot(\Delta_1,\Delta_2) = &\int\left(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_1 + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\Delta_1 + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^+\cdot\Delta_1\right)d\Delta_2(\boldsymbol{z}) \\
&+\int\left(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\Delta_2 + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\Delta_2 + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^+\cdot\Delta_2\right)d\Delta_1(\boldsymbol{z}) \\
&+\mathsf{BB}\cdot\left(\partial\bar{\boldsymbol{X}}\cdot\Delta_1,\partial\bar{\boldsymbol{X}}\cdot\Delta_2\right) + \mathsf{CC}\left(\partial\bar{\boldsymbol{X}}_\Lambda\cdot\Delta_2\right)\partial\bar{\Lambda}\cdot\Delta_1 + \mathsf{CC}\left(\partial\bar{\boldsymbol{X}}_\Lambda\cdot\Delta_1\right)\partial\bar{\Lambda}\cdot\Delta_2.
\end{aligned} \tag{90}
$$

B.6.2 Derivatives w.r.t. shocks

Next we proceed by taking derivatives w.r.t. σ and then use the method of undetermined coefficients to find the derivatives associated with each pair of shocks. Solving for $\bar{\boldsymbol{x}}_{\varepsilon\varepsilon}, \bar{\boldsymbol{x}}_{\varepsilon\boldsymbol{\Theta}}, \bar{\boldsymbol{x}}_{\varepsilon\Lambda}, \bar{\boldsymbol{x}}_{\varepsilon\sigma}$ and $\bar{\boldsymbol{x}}_{\varepsilon\boldsymbol{\mathcal{E}}}$ are trivial as these interactions do not effect aggregates so we omit those formulas them for brevity.

For $\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\Lambda}$ we see (after defining $\bar{\boldsymbol{x}}^+_{\boldsymbol{\mathcal{E}}} = \bar{\boldsymbol{x}}_{\boldsymbol{\Theta}} + \bar{\boldsymbol{x}}_{\boldsymbol{z}}\bar{\boldsymbol{z}}_{\boldsymbol{\mathcal{E}}} + \partial\bar{\boldsymbol{x}}\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_{\boldsymbol{\mathcal{E}}}$)

$$
\begin{aligned}
0 =&\mathsf{F}_{\boldsymbol{x}}\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\Lambda} + \mathsf{F}_{\boldsymbol{X}}\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda} + \mathsf{F}_{\boldsymbol{x}+}\left(\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}\Lambda}\bar{\Lambda}_\Lambda + \bar{\boldsymbol{x}}_{\boldsymbol{z}\Lambda}\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}\bar{\Lambda}_\Lambda + \partial\bar{\boldsymbol{x}}_\Lambda\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}}\bar{\Lambda}_\Lambda + \bar{\boldsymbol{x}}_{\boldsymbol{z}}\mathsf{p}\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\Lambda} + \partial\bar{\boldsymbol{x}}\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}\Lambda}\right) \\
&+\mathsf{F}_{\boldsymbol{x}+}\bar{\boldsymbol{x}}_\Lambda\mathsf{P}\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda} + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}-}\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}}\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{X}}\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{X}}_\Lambda) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}+}\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda) + \mathsf{F}_{\boldsymbol{x}\Lambda}\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}} \\
&+\mathsf{F}_{\boldsymbol{X}\boldsymbol{x}-}\cdot(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda) + \mathsf{F}_{\boldsymbol{X}\boldsymbol{x}}\cdot(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda) + \mathsf{F}_{\boldsymbol{X}\boldsymbol{x}+}\cdot(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda) + \mathsf{F}_{\boldsymbol{X}\Lambda}\bar{\boldsymbol{X}}_\Lambda + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}-}\cdot(\bar{\boldsymbol{x}}^+_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda) \\
&+\mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}}\cdot(\bar{\boldsymbol{x}}^+_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{X}}\cdot(\bar{\boldsymbol{x}}^+_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{X}}_\Lambda) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}+}\cdot(\bar{\boldsymbol{x}}^+_{\boldsymbol{\mathcal{E}}},\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda) + \mathsf{F}_{\boldsymbol{x}+\Lambda}(\bar{\boldsymbol{x}}^+_{\boldsymbol{\mathcal{E}}}) \\
&+\mathsf{F}_{\boldsymbol{\mathcal{E}}\boldsymbol{x}-}\bar{\boldsymbol{x}}_\Lambda + \mathsf{F}_{\boldsymbol{\mathcal{E}}\boldsymbol{x}}\bar{\boldsymbol{x}}_\Lambda + \mathsf{F}_{\boldsymbol{\mathcal{E}}\boldsymbol{X}}\bar{\boldsymbol{X}}_\Lambda + \mathsf{F}_{\boldsymbol{\mathcal{E}}\boldsymbol{x}+}\bar{\boldsymbol{x}}_\Lambda\bar{\Lambda}_\Lambda + \mathsf{F}_{\boldsymbol{\mathcal{E}}\Lambda}
\end{aligned}
$$

Most of these terms are already known, but there are a few that are new and need to be computed: $\partial\bar{\boldsymbol{x}}\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}\Lambda}$ and $\partial\bar{\boldsymbol{x}}_\Lambda\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}}$. For the first, $\partial\bar{\boldsymbol{x}}\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}\Lambda} = \mathsf{C}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}\Lambda} \equiv \mathsf{C}(\boldsymbol{z})\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\Lambda}$. For the second, we have

$$
\partial\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z})\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}} = \mathsf{Y}(\boldsymbol{z})^{-1}\mathsf{z}_1(\boldsymbol{z})\partial\bar{\boldsymbol{X}}_\Lambda\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}} + \mathsf{Y}(\boldsymbol{z})^{-1}\mathsf{z}_2(\boldsymbol{z})\boldsymbol{X}'_{\boldsymbol{\mathcal{E}}}
$$

where

$$
\begin{aligned}
\partial\bar{\boldsymbol{X}}_\Lambda\cdot\bar{\Omega}_{\boldsymbol{\mathcal{E}}} &= \mathsf{Y}^{-1}\mathsf{Z}\boldsymbol{X}'_{\boldsymbol{\mathcal{E}}} + \mathsf{Y}^{-1}\int\left(\mathsf{R}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda\right)\bar{\omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\boldsymbol{z} \\
&= \mathsf{Y}^{-1}\mathsf{Z}\boldsymbol{X}'_{\boldsymbol{\mathcal{E}}} + \mathsf{Y}^{-1}\int\Bigg(\frac{\partial}{\partial\boldsymbol{z}}\mathsf{R}_\Lambda(\boldsymbol{z}) + \frac{\partial}{\partial\boldsymbol{z}}\mathsf{R}_\Lambda(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{z}) + \mathsf{R}_\Lambda(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}\boldsymbol{z}}(\boldsymbol{z}) \\
&\qquad + \frac{\partial}{\partial\boldsymbol{z}}\mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda + \frac{d}{d\boldsymbol{z}}\mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\Lambda\bar{\boldsymbol{z}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\Bigg)\omega(\boldsymbol{z})d\boldsymbol{z}
\end{aligned}
$$

where the last term is easily computable from known terms.[23] Thus, we can show that the derivative $\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\Lambda}$ solves the following system of equations

$$\mathsf{DD}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\Lambda} = \mathsf{EE}(\boldsymbol{z})\begin{bmatrix} I & \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda} & \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\Lambda} \end{bmatrix}. \tag{91}$$

To determine $\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda}$ and $\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\Lambda}$, proceed in the same manner as the first-order terms. Recall we had,

$$\begin{aligned}\tilde{\Omega}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\Omega,\Lambda)(\boldsymbol{y}) = -\int \sum_i \delta(\tilde{\boldsymbol{z}}^i\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) - \boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{\boldsymbol{z}}^j\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) \leq \boldsymbol{y}^j\right)\\ \times \tilde{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega\right)d\mathrm{Pr}(\varepsilon)d\Omega(\boldsymbol{z}),\end{aligned}$$

and thus

$$\begin{aligned}\tilde{\Omega}_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\Omega,\Lambda)(\boldsymbol{y}) = &-\int \sum_i \delta(\tilde{\boldsymbol{z}}^i\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) - \boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{\boldsymbol{z}}^j\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) \leq \boldsymbol{y}^j\right)\\ &\times \tilde{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}\Lambda}\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right)d\mathrm{Pr}(\varepsilon)d\Omega(\boldsymbol{z})\\ &-\int\Bigg(\sum_i \delta'(\tilde{\boldsymbol{z}}^i\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) - \boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{\boldsymbol{z}}^j\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) \leq \boldsymbol{y}^j\right)\\ &\qquad\times \tilde{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right)\tilde{\boldsymbol{z}}^i_{\Lambda}\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right)\Bigg)d\mathrm{Pr}\;(\varepsilon)\;d\Omega(\boldsymbol{z})\\ &+\int\Bigg(\sum_i\sum_{j\neq i}\delta(\tilde{\boldsymbol{z}}^i\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) - \boldsymbol{y}^i)\delta(\tilde{\boldsymbol{z}}^j\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) - \boldsymbol{y}^j)\\ &\prod_{k\neq i,j}\iota\left(\tilde{\boldsymbol{z}}^k\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right) \leq \boldsymbol{y}^k\right)\tilde{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}}\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right)\\ &\qquad\times \tilde{\boldsymbol{z}}^i_{\Lambda}\left(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda\right)\Bigg)d\mathrm{Pr}\;(\varepsilon)\;d\Omega(\boldsymbol{z})\end{aligned}$$

Evaluating this term at $\sigma = 0$ and exploiting $\bar{\boldsymbol{z}}_{\Lambda}(\boldsymbol{z}) = 0$ we have

$$\bar{\Omega}_{\boldsymbol{\mathcal{E}}\Lambda} = -\sum_i\int\delta(\boldsymbol{z}^i - \boldsymbol{y}^i)\prod_{j\neq i}\iota(\boldsymbol{z}^j \leq \boldsymbol{y}^j)\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{z})d\Omega(\boldsymbol{z})$$

[23]For compactness we have not expanded terms like $\frac{\partial}{\partial \boldsymbol{z}}\mathsf{R}_{\Lambda}(\boldsymbol{z})$, but this is easily accomplished. For example $\frac{\partial}{\partial \boldsymbol{z}}\mathsf{R}_{\Lambda}(\boldsymbol{z}) = \mathsf{R}_{\Lambda\boldsymbol{z}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{\Theta}\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\Lambda}(\boldsymbol{z})$.

and thus

$$\bar{\omega}_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{y}) = -\sum_i \frac{\partial}{\partial \boldsymbol{y}^i} \int \prod_j \delta(\boldsymbol{z}^j - \boldsymbol{y}^j) \bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{z}) d\Omega(\boldsymbol{z}) = -\sum_i \frac{\partial}{\partial \boldsymbol{y}^i} \left(\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{y}) \omega(\boldsymbol{y}) \right)$$

and hence

$$\begin{aligned}
\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\Lambda} &= -\mathsf{D}^{-1} \int \mathsf{R}(\boldsymbol{z}) \sum_i \frac{\partial}{\partial \boldsymbol{z}^i} \left(\bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{z}) \omega(\boldsymbol{z}) \right) d\boldsymbol{z} \\
&= \mathsf{D}^{-1} \int \sum_i \left(\frac{\partial}{\partial \boldsymbol{z}^i} \mathsf{R}(\boldsymbol{z}) \right) \bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{z}) \omega(\boldsymbol{z}) d\boldsymbol{z} \\
&= \mathsf{D}^{-1} \int \sum_i \left(\mathsf{R}_{\boldsymbol{z}^i}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z}) \bar{\boldsymbol{x}}_{\boldsymbol{z}^i}(\boldsymbol{z}) \right) \bar{\boldsymbol{z}}^i_{\boldsymbol{\mathcal{E}}\Lambda} d\Omega(\boldsymbol{z}) \\
&= \mathsf{D}^{-1} \int \left(\mathsf{R}_{\boldsymbol{z}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z}) \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}) \right) \mathsf{q} \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{z}) d\Omega(\boldsymbol{z})
\end{aligned}$$

which combined with the second derivative of R w.r.t. $\boldsymbol{\mathcal{E}}\Lambda$ (defining $\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}} = \bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} + \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{X}}_\Lambda \bar{\Lambda}_{\boldsymbol{\mathcal{E}}}$)

$$\begin{aligned}
\int \Bigg(& \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z}) \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z}) \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda} + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z}) \left(\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}\Lambda} \bar{\Lambda}_\Lambda + \partial \bar{\boldsymbol{X}}_\Lambda \cdot \bar{\Omega}_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{X}}_{\Lambda\Lambda} \bar{\Lambda}_\Lambda \bar{\Lambda}_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\Lambda} + \bar{\boldsymbol{X}}_\Lambda \mathsf{P} \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda} \right) \\
&+ \mathsf{R}_{\boldsymbol{x}\boldsymbol{x}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) \right) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), \bar{\boldsymbol{X}}_\Lambda \right) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}+} \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), \bar{\boldsymbol{X}}_\Lambda \bar{\Lambda}_\Lambda \right) + \mathsf{R}_{\boldsymbol{x}\Lambda}(\boldsymbol{z}) \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \\
&+ \mathsf{R}_{\boldsymbol{X}\boldsymbol{x}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) \right) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}_\Lambda \right) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}+}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}_\Lambda \bar{\Lambda}_\Lambda \right) + \mathsf{R}_{\boldsymbol{X}\Lambda}(\boldsymbol{z}) \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} \\
&+ \mathsf{R}_{\boldsymbol{X}+\boldsymbol{x}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) \right) + \mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}_\Lambda \right) + \mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}+}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}_\Lambda \bar{\Lambda}_\Lambda \right) + \mathsf{R}_{\boldsymbol{X}+\Lambda}(\boldsymbol{z}) \bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}} \\
&+ \mathsf{R}_{\boldsymbol{\mathcal{E}}\boldsymbol{x}}(\boldsymbol{z}) \bar{\boldsymbol{x}}_\Lambda(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{\mathcal{E}}\boldsymbol{X}}(\boldsymbol{z}) \bar{\boldsymbol{X}}_\Lambda + \mathsf{R}_{\boldsymbol{\mathcal{E}}\boldsymbol{X}+}(\boldsymbol{z}) \bar{\boldsymbol{X}}_\Lambda \bar{\Lambda}_\Lambda + \mathsf{R}_{\boldsymbol{\mathcal{E}}\Lambda}(\boldsymbol{z}) \Bigg) d\Omega(\boldsymbol{z}) = 0
\end{aligned}$$

gives a linear relationship

$$\mathsf{FF} \left[\begin{array}{cc} \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda} & \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\Lambda} \end{array} \right] = \mathsf{GG} \tag{92}$$

to solve for $\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\Lambda}$. A similar approach can be used to obtain expressions for $\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\boldsymbol{\Theta}}(\boldsymbol{z})$, $\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\boldsymbol{\Theta}}$, $\bar{\boldsymbol{x}}_{\sigma\boldsymbol{\Theta}}(\boldsymbol{z})$, $\bar{\boldsymbol{X}}_{\sigma\boldsymbol{\Theta}}$, $\bar{\boldsymbol{x}}_{\sigma\Lambda}(\boldsymbol{z})$, and $\bar{\boldsymbol{X}}_{\sigma\Lambda}$, which take the same form.

For $\bar{\boldsymbol{x}}_{\mathcal{E}\mathcal{E}}$ we see

$$
\begin{aligned}
0 =& \mathsf{F}_{\boldsymbol{x}}\bar{\boldsymbol{x}}_{\mathcal{E}\mathcal{E}} + \mathsf{F}_{\boldsymbol{X}}\bar{\boldsymbol{X}}_{\mathcal{E}\mathcal{E}} + \mathsf{F}_{\boldsymbol{x}+}\Big(\bar{\boldsymbol{x}}_{\boldsymbol{z}}\mathsf{p}\bar{\boldsymbol{x}}_{\mathcal{E}\mathcal{E}} + \partial\bar{\boldsymbol{x}}\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}} + \bar{\boldsymbol{x}}_{\Lambda}\mathsf{P}\bar{\boldsymbol{X}}_{\mathcal{E}\mathcal{E}} + \bar{\boldsymbol{x}}_{\boldsymbol{\Theta}\boldsymbol{\Theta}} + \bar{\boldsymbol{x}}_{\boldsymbol{\Theta}\boldsymbol{z}}\cdot(I,\bar{\boldsymbol{z}}_{\mathcal{E}}) + \partial\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}\cdot\big(I,\bar{\Omega}_{\mathcal{E}}\big) \\
&+ \bar{\boldsymbol{x}}_{\boldsymbol{\Theta}\Lambda}\cdot\big(I,\bar{\Lambda}_{\mathcal{E}}\big) + \bar{\boldsymbol{x}}_{\boldsymbol{z}\boldsymbol{z}}\cdot(\bar{\boldsymbol{z}}_{\mathcal{E}},\bar{\boldsymbol{z}}_{\mathcal{E}}) + \bar{\boldsymbol{x}}_{\boldsymbol{z}\boldsymbol{\Theta}}\cdot(\bar{\boldsymbol{z}}_{\mathcal{E}},I) + \partial\bar{\boldsymbol{x}}_{\boldsymbol{z}}\cdot\big(\bar{\boldsymbol{z}}_{\mathcal{E}},\bar{\Omega}_{\mathcal{E}}\big) + \bar{\boldsymbol{x}}_{\boldsymbol{z}\Lambda}\cdot\big(\bar{\boldsymbol{z}}_{\mathcal{E}},\bar{\Lambda}_{\mathcal{E}}\big) + \partial\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}\cdot\big(\bar{\Omega}_{\mathcal{E}},I\big) \\
&+ \partial\bar{\boldsymbol{x}}_{\boldsymbol{z}}\cdot\big(\bar{\Omega}_{\mathcal{E}},\bar{\boldsymbol{z}}_{\mathcal{E}}\big) + \partial\bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}\cdot\big(\bar{\Omega}_{\mathcal{E}},I\big) + \partial\bar{\boldsymbol{x}}_{\boldsymbol{z}}\cdot\big(\bar{\Omega}_{\mathcal{E}},\bar{\boldsymbol{z}}_{\mathcal{E}}\big) + \partial^2\bar{\boldsymbol{x}}\cdot(\bar{\Omega}_{\mathcal{E}},\bar{\Omega}_{\mathcal{E}}) + \partial\bar{\boldsymbol{x}}_{\Lambda}\cdot\big(\bar{\Omega}_{\mathcal{E}},\bar{\Lambda}_{\mathcal{E}}\big) \\
&+ \bar{\boldsymbol{x}}_{\Lambda\boldsymbol{\Theta}}\cdot\big(\bar{\Lambda}_{\mathcal{E}},I\big) + \bar{\boldsymbol{x}}_{\Lambda\boldsymbol{z}}\cdot\big(\bar{\Lambda}_{\mathcal{E}},\bar{\boldsymbol{z}}_{\mathcal{E}}\big) + \partial\bar{\boldsymbol{x}}_{\Lambda}\cdot\big(\Lambda_{\mathcal{E}},\bar{\Omega}_{\mathcal{E}}\big) + \bar{\boldsymbol{x}}_{\Lambda\Lambda}\cdot(\Lambda_{\mathcal{E}},\Lambda_{\mathcal{E}})\Big) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}}\cdot(\bar{\boldsymbol{x}}_{\mathcal{E}},\bar{\boldsymbol{x}}_{\mathcal{E}}) \\
&+ \mathsf{F}_{\boldsymbol{x}\boldsymbol{X}}\cdot\big(\bar{\boldsymbol{x}}_{\mathcal{E}},\bar{\boldsymbol{X}}_{\mathcal{E}}\big) + \mathsf{F}_{\boldsymbol{x}\boldsymbol{x}+}\cdot\big(\bar{\boldsymbol{x}}_{\mathcal{E}},\bar{\boldsymbol{x}}^{+}_{\mathcal{E}}\big) + \mathsf{F}_{\boldsymbol{x}\mathcal{E}}\cdot(\bar{\boldsymbol{x}}_{\mathcal{E}},I) + \mathsf{F}_{\boldsymbol{X}\boldsymbol{x}}\cdot\big(\bar{\boldsymbol{X}}_{\mathcal{E}},\bar{\boldsymbol{x}}_{\mathcal{E}}\big) + \mathsf{F}_{\boldsymbol{X}\boldsymbol{X}}\cdot\big(\bar{\boldsymbol{X}}_{\mathcal{E}},\bar{\boldsymbol{X}}_{\mathcal{E}}\big) \\
&+ \mathsf{F}_{\boldsymbol{X}\boldsymbol{x}+}\cdot\big(\bar{\boldsymbol{X}}_{\mathcal{E}},\bar{\boldsymbol{x}}^{+}_{\mathcal{E}}\big) + \mathsf{F}_{\boldsymbol{X}\mathcal{E}}\cdot\big(\bar{\boldsymbol{X}}_{\mathcal{E}},I\big) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}}\cdot\big(\bar{\boldsymbol{x}}^{+}_{\mathcal{E}},\bar{\boldsymbol{x}}_{\mathcal{E}}\big) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{X}}\cdot\big(\bar{\boldsymbol{x}}^{+}_{\mathcal{E}},\bar{\boldsymbol{X}}_{\mathcal{E}}\big) + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}+}\cdot\big(\bar{\boldsymbol{x}}^{+}_{\mathcal{E}},\bar{\boldsymbol{x}}^{+}_{\mathcal{E}}\big) \\
&+ \mathsf{F}_{\boldsymbol{x}+\mathcal{E}}\cdot\big(\bar{\boldsymbol{x}}^{+}_{\mathcal{E}},I\big) + \mathsf{F}_{\mathcal{E}\boldsymbol{x}}\cdot(I,\bar{\boldsymbol{x}}_{\mathcal{E}}) + \mathsf{F}_{\mathcal{E}\boldsymbol{X}}\cdot\big(I,\bar{\boldsymbol{X}}_{\mathcal{E}}\big) + \mathsf{F}_{\mathcal{E}\boldsymbol{x}+}\cdot\big(I,\bar{\boldsymbol{x}}^{+}_{\mathcal{E}}\big) + \mathsf{F}_{\mathcal{E}\mathcal{E}}
\end{aligned}
$$

Once again, almost all of these terms are already known with the exception of $\partial^2\bar{\boldsymbol{x}}\cdot(\bar{\Omega}_{\mathcal{E}},\bar{\Omega}_{\mathcal{E}})$ which is computable from the expressions for $\partial^2\bar{\boldsymbol{X}}$ above and $\partial\bar{\boldsymbol{x}}\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}}$ which equals

$$\partial\bar{\boldsymbol{x}}(\boldsymbol{z})\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}} = \mathsf{C}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}} = \mathsf{C}(\boldsymbol{z})\bar{\boldsymbol{X}}'_{\mathcal{E}\mathcal{E}}.$$

Thus, we have that

$$\mathsf{H}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\mathcal{E}\mathcal{E}}(\boldsymbol{z}) = \mathsf{HH}(\boldsymbol{z})\begin{bmatrix} I & \bar{\boldsymbol{X}}_{\mathcal{E}\mathcal{E}} & \bar{\boldsymbol{X}}'_{\mathcal{E}\mathcal{E}}\end{bmatrix}^{\mathsf{T}}, \tag{93}$$

where $\mathsf{H}(\boldsymbol{z})$ is the same as in the first-order expansion equation (80) and $\mathsf{HH}(\boldsymbol{z})$ is computable from the first-order terms.

To solve for $\bar{\boldsymbol{X}}_{\mathcal{E}\mathcal{E}}$ and $\bar{\boldsymbol{X}}'_{\mathcal{E}\mathcal{E}}$ we first need $\bar{\Omega}_{\mathcal{E}\mathcal{E}}$ which we find by differentiating the law of motion for Ω to get

$$
\begin{aligned}
\tilde{\Omega}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\Omega,\Lambda)(\boldsymbol{y}) = & -\int \sum_i \delta(\tilde{z}^i(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{z}^j(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\leq \boldsymbol{y}^j\right) \\
& \quad \times \tilde{z}^i_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\, d\Pr(\varepsilon)d\Omega(\boldsymbol{z}) \\
& -\int \Bigg(\sum_i \delta'(\tilde{z}^i(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{z}^j(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\leq \boldsymbol{y}^j\right) \\
& \qquad \times \tilde{z}^i_{\boldsymbol{\mathcal{E}}}(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\,\tilde{z}^i_{\boldsymbol{\mathcal{E}}}(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\Bigg) d\Pr(\varepsilon)\; d\Omega(\boldsymbol{z}) \\
& +\int \Bigg(\sum_i\sum_{j\neq i}\delta(\tilde{z}^i(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^i)\delta(\tilde{z}^j(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^j) \\
& \qquad \times \prod_{k\neq i,j}\iota\left(\tilde{z}^k(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\leq \boldsymbol{y}^k\right)\tilde{z}^j_{\boldsymbol{\mathcal{E}}}(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda) \\
& \qquad \times \tilde{z}^i_{\boldsymbol{\mathcal{E}}}(\varepsilon,\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\, d\Pr(\varepsilon)\; d\Omega(\boldsymbol{z})\Bigg).
\end{aligned}
$$

Evaluated at $\sigma = 0$, this becomes

$$
\begin{aligned}
\bar{\Omega}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{y}) = & -\int\sum_i \delta(\boldsymbol{z}^i-\boldsymbol{y}^i)\prod_{j\neq i}\iota(\boldsymbol{z}^j-\boldsymbol{y}^j)\bar{z}^i_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) \\
& -\int\sum_i \delta'(\boldsymbol{z}^i-\boldsymbol{y}^i)\prod_{j\neq i}\iota(\boldsymbol{z}^j-\boldsymbol{y}^j)\left[\bar{z}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\right]^2 d\Omega(\boldsymbol{z}) \\
& +\int\sum_i \delta(\boldsymbol{z}^i-\boldsymbol{y}^i)\sum_{j\neq i}\delta(\boldsymbol{z}^j-\boldsymbol{y}^j)\prod_{k\neq i,j}\iota(\boldsymbol{z}^k-\boldsymbol{y}^k)\bar{z}^j_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\bar{z}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z})
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\omega}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{y}) = \frac{\partial^{n_z}}{\partial\boldsymbol{y}^1\partial\boldsymbol{y}^2\cdots\partial\boldsymbol{y}^{n_z}}\bar{\Omega}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{y}) = & -\sum_i\frac{\partial}{\partial\boldsymbol{y}^i}\int\prod_j\delta(\boldsymbol{z}^j-\boldsymbol{y}^j)\bar{z}^i_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega(\boldsymbol{z}) \\
& -\sum_i\frac{\partial}{\partial\boldsymbol{y}^i}\int\delta'(\boldsymbol{z}^i-\boldsymbol{y}^i)\prod_{j\neq i}\delta(\boldsymbol{z}^j-\boldsymbol{y}^j)\left(\bar{z}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\right)^2 d\Omega(\boldsymbol{z}) \\
& +\sum_i\sum_{j\neq i}\frac{\partial^2}{\partial\boldsymbol{y}^i\partial\boldsymbol{y}^j}\int\prod_j\delta(\boldsymbol{z}^j-\boldsymbol{y}^j)\bar{z}^j_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\bar{z}^i_{\boldsymbol{\mathcal{E}}}d\Omega(\boldsymbol{z}) \\
= & -\sum_i\frac{\partial}{\partial\boldsymbol{y}^i}\left(\bar{z}^i_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{y})\omega(\boldsymbol{y})\right)+\sum_i\sum_j\frac{\partial^2}{\partial\boldsymbol{y}^i\partial\boldsymbol{y}^j}\left(\bar{z}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y})\bar{z}^j_{\boldsymbol{\mathcal{E}}}(\boldsymbol{y})\omega(\boldsymbol{y})\right).
\end{aligned}
$$

Thus,

$$\bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}} = \int \mathsf{R}(\boldsymbol{z}) \left(-\sum_i \frac{\partial}{\partial \boldsymbol{z}^i} \left(\bar{z}^i_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\omega(\boldsymbol{z}) \right) + \sum_i \sum_j \frac{\partial^2}{\partial \boldsymbol{z}^i \partial \boldsymbol{z}^j} \left(\bar{z}^i_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \bar{z}^j_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\omega(\boldsymbol{z}) \right) \right) d\boldsymbol{z}$$
$$= \int \left(\mathsf{R}_{\boldsymbol{z}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z}) \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}) \right) \mathsf{p} \bar{\boldsymbol{x}}^i_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{z})\omega(\boldsymbol{z}) d\boldsymbol{z}$$
$$+ \int \Big(\mathsf{R}_{\boldsymbol{z}\boldsymbol{z}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{z}}(\boldsymbol{z}) \cdot (I, \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})) + \mathsf{R}_{\boldsymbol{z}\boldsymbol{x}}(\boldsymbol{z}) \cdot (\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}), I) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{x}}(\boldsymbol{z}) \cdot (\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}), \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})) \Big)$$
$$\cdot \left(\mathsf{p} \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), \mathsf{p} \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \right) \omega(\boldsymbol{z}) d\boldsymbol{z}$$

which when combined with the second derivative of R w.r.t $\mathcal{E}\mathcal{E}$

$$0 = \int \Bigg(\mathsf{R}_{\boldsymbol{X}^+}(\boldsymbol{z}) \Big(\bar{\boldsymbol{X}}_{\boldsymbol{\Theta}\boldsymbol{\Theta}} + \partial \bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} \cdot \left(I, \bar{\Omega}_{\boldsymbol{\mathcal{E}}} \right) + \bar{\boldsymbol{X}}_{\boldsymbol{\Theta}\Lambda} \cdot \left(I, \bar{\Lambda}_{\boldsymbol{\mathcal{E}}} \right) + \partial \bar{\boldsymbol{X}}_{\boldsymbol{\Theta}} \cdot \left(\bar{\Omega}_{\boldsymbol{\mathcal{E}}}, I \right) + \partial^2 \bar{\boldsymbol{X}} \cdot \left(\bar{\Omega}_{\boldsymbol{\mathcal{E}}}, \bar{\Omega}_{\boldsymbol{\mathcal{E}}} \right)$$
$$+ \partial \bar{\boldsymbol{X}}_{\Lambda} \cdot \left(\bar{\Omega}_{\boldsymbol{\mathcal{E}}}, \Lambda_{\boldsymbol{\mathcal{E}}} \right) + \bar{\boldsymbol{X}}_{\Lambda\boldsymbol{\Theta}} \cdot \left(\bar{\Lambda}_{\boldsymbol{\mathcal{E}}}, I \right) + \partial \bar{\boldsymbol{X}}_{\Lambda} \cdot \left(\bar{\Lambda}_{\boldsymbol{\mathcal{E}}}, I \right) + \bar{\boldsymbol{X}}_{\Lambda\Lambda} \bar{\Lambda}^2_{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{X}}_{\Lambda} \mathsf{P} \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}} \Big)$$
$$+ \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z}) \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z}) \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}} + \mathsf{R}_{\boldsymbol{x}\boldsymbol{x}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \right) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} \right)$$
$$+ \mathsf{R}_{\boldsymbol{x}\boldsymbol{X}^+}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), \bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}} \right) + \mathsf{R}_{\boldsymbol{x}\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}), I \right) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{x}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \right)$$
$$+ \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} \right) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{X}^+}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}} \right) + \mathsf{R}_{\boldsymbol{X}\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}, I \right)$$
$$+ \mathsf{R}_{\boldsymbol{X}^+\boldsymbol{x}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \right) + \mathsf{R}_{\boldsymbol{X}^+\boldsymbol{X}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} \right) + \mathsf{R}_{\boldsymbol{X}^+\boldsymbol{X}^+}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}}, \bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}} \right)$$
$$+ \mathsf{R}_{\boldsymbol{X}^+\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \cdot \left(\bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}}, I \right) + \mathsf{R}_{\boldsymbol{\mathcal{E}}\boldsymbol{x}}(\boldsymbol{z}) \cdot \left(I, \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \right) + \mathsf{R}_{\boldsymbol{\mathcal{E}}\boldsymbol{X}}(\boldsymbol{z}) \cdot \left(I, \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}} \right)$$
$$+ \mathsf{R}_{\boldsymbol{\mathcal{E}}\boldsymbol{X}^+}(\boldsymbol{z}) \cdot \left(I, \bar{\boldsymbol{X}}^+_{\boldsymbol{\mathcal{E}}} \right) + \mathsf{R}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}}(\boldsymbol{z}) \Bigg) d\Omega(\boldsymbol{z})$$

gives a system of linear equations of the form

$$\mathsf{K} \cdot \begin{bmatrix} \bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}} & \bar{\boldsymbol{X}}'_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}} \end{bmatrix}^{\mathsf{T}} = \mathsf{II}. \tag{94}$$

A similar approach can be used to obtain expressions for $\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}\sigma}(\boldsymbol{z})$ and $\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}\sigma}$ which take the same form.

Finally, for the second-order expansion we need the effect of the presence of risk. Differentiation of F gives, after defining $\bar{\boldsymbol{x}}_\sigma^+ = \bar{\boldsymbol{x}}_z \bar{\boldsymbol{z}}_\sigma + \partial\bar{\boldsymbol{x}} \cdot \bar{\Omega}_\sigma + \bar{\boldsymbol{x}}_\Lambda \bar{\Lambda}_\sigma + \bar{\boldsymbol{x}}_\sigma$,

$$
\begin{aligned}
0 =& \mathsf{F}_{\boldsymbol{x}} \bar{\boldsymbol{x}}_{\sigma\sigma} + \mathsf{F}_{\boldsymbol{X}} \bar{\boldsymbol{X}}_{\sigma\sigma} + \mathsf{F}_{\boldsymbol{x}+} \left(\mathbb{E}\left[\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon\varepsilon}} \cdot (\varepsilon, \varepsilon) + \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}\mathcal{E}}} \cdot (\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}) \right] + \bar{\boldsymbol{x}}_{\boldsymbol{z}} \mathsf{p} \bar{\boldsymbol{x}}_{\sigma\sigma} + \partial\bar{\boldsymbol{x}} \cdot \bar{\Omega}_{\sigma\sigma} + \bar{\boldsymbol{x}}_\Lambda \mathsf{P} \bar{\boldsymbol{X}}_{\sigma\sigma} \right) \\
& + \mathsf{F}_{\boldsymbol{x}+} \Bigg(\bar{\boldsymbol{x}}_{\boldsymbol{zz}} \cdot (\bar{\boldsymbol{z}}_\sigma, \bar{\boldsymbol{z}}_\sigma) + 2\partial\bar{\boldsymbol{x}}_{\boldsymbol{z}} \cdot (\bar{\Omega}_\sigma, \bar{\boldsymbol{z}}_\sigma) + 2\bar{\boldsymbol{x}}_{\boldsymbol{z}\Lambda} \cdot (\bar{\boldsymbol{z}}_\sigma, \bar{\Lambda}_\sigma) + 2\bar{\boldsymbol{x}}_{\boldsymbol{z}\sigma} \bar{\boldsymbol{z}}_\sigma + \partial^2 \bar{\boldsymbol{x}} \cdot (\bar{\Omega}_\sigma, \bar{\Omega}_\sigma) \\
& + 2\partial\bar{\boldsymbol{x}}_\Lambda \cdot (\bar{\Omega}_\sigma, \bar{\Lambda}_\sigma) + 2\partial\bar{\boldsymbol{x}}_\sigma \cdot \bar{\Omega}_\sigma + \bar{\boldsymbol{x}}_{\Lambda\Lambda} \cdot (\bar{\Lambda}_\sigma, \bar{\Lambda}_\sigma) + 2\bar{\boldsymbol{x}}_{\Lambda\sigma} \bar{\Lambda}_\sigma + \bar{\boldsymbol{x}}_{\sigma\sigma} \Bigg) + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}-} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma) \\
& + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{X}} (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{X}}_\sigma) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}+} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{x}-\sigma} \bar{\boldsymbol{x}}_\sigma \\
& + \mathsf{F}_{\boldsymbol{xx}} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma) + 2\mathsf{F}_{\boldsymbol{xX}} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{X}}_\sigma) + 2\mathsf{F}_{\boldsymbol{xx}+} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{x}\sigma} \bar{\boldsymbol{x}}_\sigma + \mathsf{F}_{\boldsymbol{XX}} \cdot (\bar{\boldsymbol{X}}_\sigma, \bar{\boldsymbol{X}}_\sigma) \\
& + 2\mathsf{F}_{\boldsymbol{Xx}+} \cdot (\bar{\boldsymbol{X}}_\sigma, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{X}\sigma} \bar{\boldsymbol{X}}_\sigma + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}+} \cdot (\bar{\boldsymbol{x}}_\sigma^+, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{x}+\sigma} \bar{\boldsymbol{x}}_\sigma^+ + \mathsf{F}_{\sigma\sigma} \\
& + \mathsf{F}_{\boldsymbol{x}-} \mathbb{E}\left[\bar{\boldsymbol{x}}_{\sigma\sigma} + \bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon\varepsilon}} \cdot (\varepsilon, \varepsilon) + \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}\mathcal{E}}} \cdot (\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}) \right] = 0.
\end{aligned}
$$

or[24]

$$
\begin{aligned}
0 =& \mathsf{F}_{\boldsymbol{x}} \bar{\boldsymbol{x}}_{\sigma\sigma} + \mathsf{F}_{\boldsymbol{X}} \bar{\boldsymbol{X}}_{\sigma\sigma} + \mathsf{F}_{\boldsymbol{x}+} \left(\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon\varepsilon}} \cdot \operatorname{var}(\boldsymbol{\varepsilon}) + \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}\mathcal{E}}} \cdot \operatorname{var}(\boldsymbol{\mathcal{E}}) + \bar{\boldsymbol{x}}_{\boldsymbol{z}} \mathsf{p} \bar{\boldsymbol{x}}_{\sigma\sigma} + \partial\bar{\boldsymbol{x}} \cdot \bar{\Omega}_{\sigma\sigma} + \bar{\boldsymbol{x}}_\Lambda \mathsf{P} \bar{\boldsymbol{X}}_{\sigma\sigma} \right) \\
& + \mathsf{F}_{\boldsymbol{x}+} \Bigg(\bar{\boldsymbol{x}}_{\boldsymbol{zz}} \cdot (\bar{\boldsymbol{z}}_\sigma, \bar{\boldsymbol{z}}_\sigma) + 2\partial\bar{\boldsymbol{x}}_{\boldsymbol{z}} \cdot (\bar{\Omega}_\sigma, \bar{\boldsymbol{z}}_\sigma) + 2\bar{\boldsymbol{x}}_{\boldsymbol{z}\Lambda} \cdot (\bar{\boldsymbol{z}}_\sigma, \bar{\Lambda}_\sigma) + 2\bar{\boldsymbol{x}}_{\boldsymbol{z}\sigma} \bar{\boldsymbol{z}}_\sigma + \partial^2 \bar{\boldsymbol{x}} \cdot (\bar{\Omega}_\sigma, \bar{\Omega}_\sigma) \\
& + 2\partial\bar{\boldsymbol{x}}_\Lambda \cdot (\bar{\Omega}_\sigma, \bar{\Lambda}_\sigma) + 2\partial\bar{\boldsymbol{x}}_\sigma \cdot \bar{\Omega}_\sigma + \bar{\boldsymbol{x}}_{\Lambda\Lambda} \cdot (\bar{\Lambda}_\sigma, \bar{\Lambda}_\sigma) + 2\bar{\boldsymbol{x}}_{\Lambda\sigma} \bar{\Lambda}_\sigma + \bar{\boldsymbol{x}}_{\sigma\sigma} \Bigg) \\
& + \mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}-} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{X}} (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{X}}_\sigma) + 2\mathsf{F}_{\boldsymbol{x}-\boldsymbol{x}+} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{x}-\sigma} \bar{\boldsymbol{x}}_\sigma \\
& + \mathsf{F}_{\boldsymbol{xx}} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma) + 2\mathsf{F}_{\boldsymbol{xX}} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{X}}_\sigma) + 2\mathsf{F}_{\boldsymbol{xx}+} \cdot (\bar{\boldsymbol{x}}_\sigma, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{x}\sigma} \bar{\boldsymbol{x}}_\sigma + \mathsf{F}_{\boldsymbol{XX}} \cdot (\bar{\boldsymbol{X}}_\sigma, \bar{\boldsymbol{X}}_\sigma) \\
& + 2\mathsf{F}_{\boldsymbol{Xx}+} \cdot (\bar{\boldsymbol{X}}_\sigma, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{X}\sigma} \bar{\boldsymbol{X}}_\sigma + \mathsf{F}_{\boldsymbol{x}+\boldsymbol{x}+} \cdot (\bar{\boldsymbol{x}}_\sigma^+, \bar{\boldsymbol{x}}_\sigma^+) + 2\mathsf{F}_{\boldsymbol{x}+\sigma} \bar{\boldsymbol{x}}_\sigma^+ + \mathsf{F}_{\sigma\sigma} \\
& + \mathsf{F}_{\boldsymbol{x}-} \left[\bar{\boldsymbol{x}}_{\sigma\sigma} + \bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon\varepsilon}} \cdot \operatorname{var}(\boldsymbol{\varepsilon}) + \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}\mathcal{E}}} \cdot \operatorname{var}(\boldsymbol{\mathcal{E}}) \right]
\end{aligned}
$$

Defining $\partial\bar{\boldsymbol{x}}(\boldsymbol{z}) \cdot \bar{\Omega}_{\sigma\sigma} = \mathsf{C}(\boldsymbol{z}) \partial\bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\sigma\sigma} \equiv \mathsf{C}(\boldsymbol{z}) \bar{\boldsymbol{X}}'_{\sigma\sigma}$, we find that $\bar{\boldsymbol{x}}_{\sigma\sigma}(\boldsymbol{z})$ solves the linear system

$$
\mathsf{JJ}(\boldsymbol{z}) \bar{\boldsymbol{x}}_{\sigma\sigma}(\boldsymbol{z}) = \mathsf{KK}(\boldsymbol{z}) \begin{bmatrix} I & \bar{\boldsymbol{X}}_{\sigma\sigma} & \bar{\boldsymbol{X}}'_{\sigma\sigma} \end{bmatrix}^{\mathsf{T}}. \tag{95}
$$

[24]Recall $\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon\varepsilon}}$ is a 3-dimensional tensor. We define $\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon\varepsilon}} \cdot \operatorname{var}(\boldsymbol{\varepsilon})$ by the contraction $\sum_{\mathsf{jk}} [\bar{\boldsymbol{x}}_{\varepsilon\varepsilon}]_{\mathsf{ijk}} \operatorname{var}(\varepsilon)_{\mathsf{jk}}$

To solve for $\bar{\boldsymbol{X}}'_{\sigma\sigma}$ we need to find $\bar{\Omega}_{\sigma\sigma}$ by differentiating the law of motion for Ω [25]

$$\tilde{\Omega}_{\sigma\sigma}(\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\Omega,\Lambda)(\boldsymbol{y}) = -\int\Bigg(\sum_i \delta(\tilde{\boldsymbol{z}}^i(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{\boldsymbol{z}}^j(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\leq\boldsymbol{y}^j\right)$$

$$\left(\tilde{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\cdot(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon})+\tilde{\boldsymbol{z}}^i_{\sigma\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\right)\Bigg)d\Pr(\boldsymbol{\varepsilon})d\Omega(\boldsymbol{z})$$

$$-\int\Bigg(\sum_i \delta'(\tilde{\boldsymbol{z}}^i(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^i)\prod_{j\neq i}\iota\left(\tilde{\boldsymbol{z}}^j(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\leq\boldsymbol{y}^j\right)$$

$$\Bigg(\tilde{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\,\boldsymbol{\varepsilon}\,\tilde{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\,\boldsymbol{\varepsilon}$$

$$+\tilde{\boldsymbol{z}}^i_{\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\,\tilde{\boldsymbol{z}}^i_{\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\Bigg)\Bigg)d\Pr(\boldsymbol{\varepsilon})\ d\Omega(\boldsymbol{z})$$

$$-\int\Bigg(\sum_i\sum_{j\neq i}\delta(\tilde{\boldsymbol{z}}^i(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^i)\delta(\tilde{\boldsymbol{z}}^j(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)-\boldsymbol{y}^j)$$

$$\prod_{k\neq i,j}\iota\left(\tilde{\boldsymbol{z}}^k(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\leq\boldsymbol{y}^k\right)\Bigg(\tilde{\boldsymbol{z}}^j_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\,\boldsymbol{\varepsilon}\,\tilde{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\,\boldsymbol{\varepsilon}$$

$$\tilde{\boldsymbol{z}}^j_{\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\,\tilde{\boldsymbol{z}}^i_{\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\mathcal{E}},\boldsymbol{\Theta},\boldsymbol{z},\Omega,\Lambda)\Bigg)\Bigg)d\Pr(\boldsymbol{\varepsilon})\ d\Omega(\boldsymbol{z})$$

Evaluating at $\sigma = 0$ we find

$$\bar{\Omega}_{\sigma\sigma}(\boldsymbol{y}) = -\int\sum_i\delta(\boldsymbol{z}^i-\boldsymbol{y}^i)\prod_{j\neq i}\iota(\boldsymbol{z}^j-\boldsymbol{y}^j)\left(\bar{\boldsymbol{z}}^i_{\sigma\sigma}(\boldsymbol{z})+\bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}}\cdot\mathrm{var}(\boldsymbol{\varepsilon})\right)d\Omega(\boldsymbol{z})$$

$$-\int\sum_i\delta'(\boldsymbol{z}^i-\boldsymbol{y}^i)\prod_{j\neq i}\iota(\boldsymbol{z}^j-\boldsymbol{y}^j)\left[\bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\boldsymbol{z})\right]^2\cdot\mathrm{var}(\boldsymbol{\varepsilon})+\left[\bar{\boldsymbol{z}}^i_{\sigma}(\boldsymbol{z})\right]^2 d\Omega(\boldsymbol{z})$$

$$+\int\sum_i\delta(\boldsymbol{z}^i-\boldsymbol{y}^i)\sum_{j\neq i}\delta(\boldsymbol{z}^j-\boldsymbol{y}^j)\prod_{k\neq i,j}\iota(\boldsymbol{z}^k-\boldsymbol{y}^k)\left(\bar{\boldsymbol{z}}^j_{\boldsymbol{\varepsilon}}(\boldsymbol{z})\bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\boldsymbol{z})\right)\cdot\mathrm{var}(\boldsymbol{\varepsilon})+\bar{\boldsymbol{z}}^j_{\sigma}(\boldsymbol{z})\bar{\boldsymbol{z}}^i_{\sigma}(\boldsymbol{z})d\Omega(\boldsymbol{z})$$

which gives

$$\bar{\omega}_{\sigma\sigma}(\boldsymbol{y}) = -\sum_i\frac{\partial}{\partial\boldsymbol{y}^i}\left(\left(\bar{\boldsymbol{z}}^i_{\sigma\sigma}(\boldsymbol{y})+\bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}}(\boldsymbol{y})\cdot\mathrm{var}(\boldsymbol{\varepsilon})\right)\omega(\boldsymbol{y})\right)$$

$$+\sum_i\sum_j\frac{\partial^2}{\partial\boldsymbol{y}^i\partial\boldsymbol{y}^j}\left(\left(\left(\bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\boldsymbol{y})\bar{\boldsymbol{z}}^j_{\boldsymbol{\varepsilon}}(\boldsymbol{y})\right)\cdot\mathrm{var}(\boldsymbol{\varepsilon})+\bar{\boldsymbol{z}}^i_{\sigma}(\boldsymbol{y})\bar{\boldsymbol{z}}^j_{\sigma}(\boldsymbol{y})\right)\omega(\boldsymbol{y})\right).$$

[25]For simplicity we drop the cross terms between σ and $\boldsymbol{\varepsilon}$ which integrate to 0 when evaluated at $\sigma = 0$.

Thus,

$$
\begin{aligned}
\bar{\boldsymbol{X}}'_{\sigma\sigma} &= \int \mathsf{R}(\boldsymbol{z})\Bigg(-\sum_i \frac{\partial}{\partial \boldsymbol{z}^i}\left((\bar{\boldsymbol{z}}^i_{\sigma\sigma}(\boldsymbol{z}) + \bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon\varepsilon}}(\boldsymbol{y})\mathrm{var}(\boldsymbol{\varepsilon}))\omega(\boldsymbol{z})\right) \\
&\quad + \sum_i\sum_j \frac{\partial^2}{\partial \boldsymbol{z}^i \partial \boldsymbol{z}^j}\left(\left(\bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon}}(\boldsymbol{z})\bar{\boldsymbol{z}}^j_{\boldsymbol{\varepsilon}}(\boldsymbol{z})\mathrm{var}(\boldsymbol{\varepsilon}) + \bar{\boldsymbol{z}}^i_{\sigma}(\boldsymbol{z})\bar{\boldsymbol{z}}^j_{\sigma}(\boldsymbol{z})\right)\omega(\boldsymbol{z})\right)\Bigg)d\boldsymbol{z} \\
&= \int \left(\mathsf{R}_{\boldsymbol{z}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})\right)\left(\bar{\boldsymbol{z}}^i_{\sigma\sigma}(\boldsymbol{z}) + \bar{\boldsymbol{z}}^i_{\boldsymbol{\varepsilon\varepsilon}}(\boldsymbol{y})\mathrm{var}(\boldsymbol{\varepsilon})\right)\omega(\boldsymbol{z})d\boldsymbol{z} \\
&\quad + \int \Bigg(\mathsf{R}_{\boldsymbol{zz}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{xz}}(\boldsymbol{z})\cdot(I, \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})) + \mathsf{R}_{\boldsymbol{zx}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}), I) + \mathsf{R}_{\boldsymbol{xx}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}), \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}))\Bigg) \\
&\qquad\qquad \cdot (\bar{\boldsymbol{z}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}), \bar{\boldsymbol{z}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}))\cdot \mathrm{var}(\boldsymbol{\varepsilon})\omega(\boldsymbol{z})d\boldsymbol{z} \\
&\quad + \int \Bigg(\mathsf{R}_{\boldsymbol{zz}}(\boldsymbol{z}) + \mathsf{R}_{\boldsymbol{xz}}(\boldsymbol{z})\cdot(I, \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z})) + \mathsf{R}_{\boldsymbol{zx}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}), I) + \mathsf{R}_{\boldsymbol{xx}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}), \bar{\boldsymbol{x}}_{\boldsymbol{z}}(\boldsymbol{z}))\Bigg) \\
&\qquad\qquad \cdot (\bar{\boldsymbol{z}}_{\sigma}(\boldsymbol{z}), \bar{\boldsymbol{z}}_{\sigma}(\boldsymbol{z}))\cdot \mathrm{var}(\boldsymbol{\varepsilon})\omega(\boldsymbol{z})d\boldsymbol{z}
\end{aligned}
$$

which combined with the second derivative of R w.r.t $\sigma\sigma$, after defining $\bar{\boldsymbol{X}}^+_\sigma = \bar{\boldsymbol{X}}'_\sigma + \bar{\boldsymbol{X}}_\Lambda\bar{\Lambda}_\sigma + \bar{\boldsymbol{X}}_\sigma$

$$
\begin{aligned}
&\int \Bigg(\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}_{\sigma\sigma}(\boldsymbol{z}) + \bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon\varepsilon}}(\boldsymbol{z})\cdot\mathrm{var}(\boldsymbol{\varepsilon})\right) + \mathsf{R}_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}_{\sigma\sigma} + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left(\bar{\boldsymbol{X}}'_{\sigma\sigma} + \bar{\boldsymbol{X}}_\Lambda \mathsf{P}\bar{\boldsymbol{X}}_{\sigma\sigma}\right) \\
&\quad + \mathsf{R}_{\boldsymbol{X}+}(\boldsymbol{z})\left(\bar{\boldsymbol{X}}_{\Lambda\Lambda}\bar{\Lambda}^2_\sigma + 2\partial\bar{\boldsymbol{X}}_\Lambda\cdot\bar{\Omega}_\sigma\bar{\Lambda}_\sigma + 2\bar{\boldsymbol{X}}_{\Lambda\sigma}\bar{\Lambda}_\sigma + \partial^2\bar{\boldsymbol{X}}\cdot(\bar{\Omega}_\sigma, \bar{\Omega}_\sigma) + 2\partial\bar{\boldsymbol{X}}_\sigma\cdot\bar{\Omega}_\sigma + \bar{\boldsymbol{X}}_{\sigma\sigma}\right) \\
&\quad + \mathsf{R}_{\boldsymbol{xx}}(z)\cdot(\bar{\boldsymbol{x}}_\sigma(z), \bar{\boldsymbol{x}}_\sigma(z)) + 2\mathsf{R}_{\boldsymbol{xX}}(z)\cdot(\bar{\boldsymbol{x}}_\sigma(z), \bar{\boldsymbol{X}}_\sigma) + 2\mathsf{R}_{\boldsymbol{xX}+}(z)\cdot(\bar{\boldsymbol{x}}_\sigma(z), \bar{\boldsymbol{X}}^+_\sigma) + \mathsf{R}_{\boldsymbol{XX}}(z)\cdot(\bar{\boldsymbol{X}}_\sigma, \bar{\boldsymbol{X}}_\sigma) \\
&\quad + 2\mathsf{R}_{\boldsymbol{XX}+}(z)\cdot(\bar{\boldsymbol{X}}_\sigma, \bar{\boldsymbol{X}}^+_\sigma) + 2\mathsf{R}_{\boldsymbol{X}+\boldsymbol{X}+}(z)\cdot(\bar{\boldsymbol{X}}^+_\sigma, \bar{\boldsymbol{X}}^+_\sigma) + \mathsf{R}_{\boldsymbol{xx}}(\boldsymbol{z})\cdot(\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}), \bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z}))\cdot\mathrm{var}(\boldsymbol{\varepsilon}) \\
&\quad + 2\left(\mathsf{R}_{\boldsymbol{x\varepsilon}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}(\boldsymbol{z})\right)\cdot\mathrm{var}(\boldsymbol{\varepsilon}) + \mathsf{R}_{\boldsymbol{\varepsilon\varepsilon}}(\boldsymbol{z})\cdot\mathrm{var}(\boldsymbol{\varepsilon})\Bigg)d\Omega(\boldsymbol{z}) = 0
\end{aligned}
$$

gives a system of linear equations of the form

$$
\mathsf{LL}\cdot\begin{bmatrix}\bar{\boldsymbol{X}}_{\sigma\sigma} & \bar{\boldsymbol{X}}'_{\sigma\sigma}\end{bmatrix}^{\mathsf{T}} = \mathsf{MM}. \tag{96}
$$

B.7 Expansion Along Path

We now demonstrate how to compute the expansion along transition path when assuming σ scales only the shocks and ρ_θ. Given the current $(\boldsymbol{\Theta}, \Lambda)$, we will assume knowledge of the non-stochastic transition path $\{\bar{\boldsymbol{\Theta}}^n, \bar{\Lambda}^n\}$ with $(\bar{\boldsymbol{\Theta}}^0, \bar{\Lambda}^0) = (\boldsymbol{\Theta}, \Lambda)$ and $(\bar{\boldsymbol{\Theta}}^N, \bar{\Lambda}^N) =$

$(\bar{\boldsymbol{\Theta}}, \bar{\Lambda})$ as well as the associated paths $\{\bar{\boldsymbol{x}}^n, \bar{\boldsymbol{X}}^n\}$.[26] Sections B.4-B.6 give the derivatives of the policy functions at $(\bar{\boldsymbol{\Theta}}^N, \bar{\Lambda}^N) = (\bar{\boldsymbol{\Theta}}, \bar{\Lambda})$. We then proceed by backward induction, assuming that all the derivatives of $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{X}}$ are known at the point $(0, \bar{\boldsymbol{\Theta}}^{n+1}, \Omega, \bar{\Lambda}^{n+1}; 0)$ and computing derivatives evaluated at $(0, \bar{\boldsymbol{\Theta}}^n, \Omega, \bar{\Lambda}^n; 0)$. We will denote derivatives evaluated at $(0, \bar{\boldsymbol{\Theta}}^n, \Omega, \bar{\Lambda}^n; 0)$ by superscript n.

Differentiating with respect to Λ,

$$\mathsf{F}^n_{\boldsymbol{x}-}(\boldsymbol{z})\bar{\boldsymbol{x}}^n_\Lambda(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}^n_\Lambda(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}+}(\boldsymbol{z})\left(\bar{\boldsymbol{x}}^{n+1}_\Lambda(\boldsymbol{z})\mathsf{P}\bar{\boldsymbol{X}}^n_\Lambda\right) + \mathsf{F}^n_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}^n_\Lambda = 0$$

and

$$\int \mathsf{R}^n_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}^n_\Lambda(\boldsymbol{z}) + \mathsf{R}^n_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}^n_\Lambda + \mathsf{R}^n_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}^{n+1}_\Lambda\mathsf{P}\bar{\boldsymbol{X}}^n_\Lambda + \mathsf{R}^n_\Lambda(\boldsymbol{z})d\Omega = 0.$$

Solving for $\bar{\boldsymbol{x}}^n_\Lambda(\boldsymbol{z})$ gives

$$\bar{\boldsymbol{x}}^n_\Lambda(\boldsymbol{z}) = -\left(\mathsf{F}^n_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}}(\boldsymbol{z})\right)^{-1}\left(\mathsf{F}^n_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}^{n+1}_\Lambda(\boldsymbol{z})\mathsf{P} + \mathsf{F}^n_{\boldsymbol{X}}(\boldsymbol{z})\right)\bar{\boldsymbol{X}}^n_\Lambda,$$

and therefore $\bar{\boldsymbol{X}}^n_\Lambda$ equals

$$-\left(\int\left[-\mathsf{R}^n_{\boldsymbol{x}}(\boldsymbol{z})\left(\mathsf{F}^n_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}}(\boldsymbol{z})\right)^{-1}\left(\mathsf{F}^n_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}^{n+1}_\Lambda(\boldsymbol{z})\mathsf{P} + \mathsf{F}^n_{\boldsymbol{X}}(\boldsymbol{z})\right) + \mathsf{R}^n_{\boldsymbol{X}}(\boldsymbol{z}) + \mathsf{R}^n_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}^{n+1}_\Lambda\mathsf{P}\right]d\Omega\right)^{-1}$$
$$\times\left(\int \mathsf{R}^n_\Lambda(\boldsymbol{z})d\Omega(\boldsymbol{z})\right).$$

The Frechet derivative w.r.t Ω is computed as follows:

$$\mathsf{F}^n_{\boldsymbol{x}-}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}^n(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}^n(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}+}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}^{n+1}(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}^{n+1}_\Lambda(\boldsymbol{z})\mathsf{P}\partial\bar{\boldsymbol{X}}^n + \mathsf{F}^n_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^n = 0$$

which gives

$$\begin{aligned}\partial\bar{\boldsymbol{x}}^n(\boldsymbol{z}) &= -\left(\mathsf{F}^n_{\boldsymbol{x}-}(\boldsymbol{z}) + \mathsf{F}^n_{\boldsymbol{x}}(\boldsymbol{z})\right)^{-1}\left[\left(\mathsf{F}^n_{\boldsymbol{x}+}(\boldsymbol{z})\bar{\boldsymbol{x}}^{n+1}_\Lambda(\boldsymbol{z})\mathsf{P} + \mathsf{F}^n_{\boldsymbol{X}}(\boldsymbol{z})\right)\partial\bar{\boldsymbol{X}}^n + \mathsf{F}^n_{\boldsymbol{x}+}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}^{n+1}(\boldsymbol{z})\right]\\ &= \sum_{j=0}^{N-n}\mathsf{C}^n_j(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^{n+j},\end{aligned}$$

where for the last expression we our knowledge that $\partial\bar{\boldsymbol{X}}^N = \partial\bar{\boldsymbol{X}}$ and $\partial\bar{\boldsymbol{x}}^N(\boldsymbol{z}) = \partial\bar{\boldsymbol{x}}(\boldsymbol{z}) = \mathsf{C}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}$ and then $\partial\bar{\boldsymbol{x}}^n(\boldsymbol{z}) = \sum_{j=0}^{N-n}\mathsf{C}^n_j(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^{n+j}$ is derived from the recursion.

[26] The fact that $\tilde{\Omega}(0, \boldsymbol{\Theta}, \Omega, \Lambda; 0) = \Omega$ for any $(\boldsymbol{\Theta}, \Omega, \Lambda)$ makes computation of these terms straightforward using a shooting algorithm.

To find $\partial\bar{\boldsymbol{X}}$ we then take the Frechet derivative of R to get

$$0=\int\left(\mathsf{R}^n_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}^n(\boldsymbol{z})\cdot\Delta+\mathsf{R}^n_{\boldsymbol{X}+}(\boldsymbol{z})\left((\partial\bar{\boldsymbol{X}}^{n+1}\cdot\boldsymbol{\Delta})+\bar{\boldsymbol{X}}^{n+1}_{\Lambda}\mathsf{P}(\partial\bar{\boldsymbol{X}}^n\cdot\boldsymbol{\Delta})\right)+\mathsf{R}^n_{\boldsymbol{X}}(\boldsymbol{z})\partial\bar{\boldsymbol{X}}^n\cdot\Delta\right)\omega(\boldsymbol{z})d\boldsymbol{z}$$
$$+\int\mathsf{R}^n(\boldsymbol{z})\delta(\boldsymbol{z})d\boldsymbol{z}.$$

Substituting for $\partial\bar{\boldsymbol{x}}^n(\boldsymbol{z})=\partial\bar{\boldsymbol{X}}^n$, we get

$$\partial\bar{\boldsymbol{X}}^n\cdot\Delta=-\left(\int\left(\mathsf{R}^n_{\boldsymbol{x}}(\boldsymbol{z})\mathsf{C}^n_0(\boldsymbol{z})+\mathsf{R}^n_{\boldsymbol{X}+}(\boldsymbol{z})\bar{\boldsymbol{X}}^{n+1}_{\Lambda}\mathsf{P}+\mathsf{R}^n_{\boldsymbol{X}}(\boldsymbol{z})\right)d\Omega(\boldsymbol{z})\right)^{-1}\int\mathsf{R}^n(\boldsymbol{z})\delta(\boldsymbol{z})+\mathsf{R}^n_{\boldsymbol{x}}(\boldsymbol{z})\partial\bar{\boldsymbol{x}}^n(\boldsymbol{z})\cdot\Delta d\boldsymbol{z}$$
$$\equiv(\mathsf{D}^n)^{-1}\left(\int\mathsf{R}^n(\boldsymbol{z})d\Delta+\sum_{j=1}^{N-n}\mathsf{E}^n_j\left(\partial\bar{\boldsymbol{X}}^{n+j}\cdot\Delta\right)\right)$$

Finally we can compute the response to shock $\boldsymbol{\mathcal{E}}$ (note we only need to do this for $n=0$)

$$\mathsf{F}^0_{\boldsymbol{x}}\bar{\boldsymbol{x}}^0_{\boldsymbol{\mathcal{E}}}+\mathsf{F}^0_{\boldsymbol{x}+}\left(\rho_{\boldsymbol{\Theta}}\bar{\boldsymbol{x}}^1_{\boldsymbol{\Theta}}+\bar{\boldsymbol{x}}^1_{\boldsymbol{z}}\mathsf{p}\bar{\boldsymbol{x}}^0_{\boldsymbol{\mathcal{E}}}+\partial\bar{\boldsymbol{x}}^1\cdot\bar{\Omega}^0_{\boldsymbol{\mathcal{E}}}+\bar{\boldsymbol{x}}^1_{\Lambda}\mathsf{P}\bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}}\right)+\mathsf{F}^0_{\boldsymbol{X}}\bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}}+\mathsf{F}^0_{\boldsymbol{\mathcal{E}}}=0.$$

Substituting for $\partial\bar{\boldsymbol{x}}^1$ we obtain, after defining $\bar{\boldsymbol{X}}^j{}_{\boldsymbol{\mathcal{E}}}=\partial\bar{\boldsymbol{X}}^j\cdot\bar{\Omega}^0_{\boldsymbol{\mathcal{E}}}$

$$\mathsf{F}^0_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})+\mathsf{F}^0_{\boldsymbol{x}+}(\boldsymbol{z})\left(\rho_{\boldsymbol{\Theta}}\bar{\boldsymbol{x}}^1_{\boldsymbol{\Theta}}(\boldsymbol{z})+\bar{\boldsymbol{x}}^1_{\boldsymbol{z}}(\boldsymbol{z})\mathsf{p}\bar{\boldsymbol{x}}^n_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})+\sum_{j=0}^{N-1}\mathsf{C}^1_j(\boldsymbol{z})\bar{\boldsymbol{X}}^{j+1}_{\boldsymbol{\mathcal{E}}}+\bar{\boldsymbol{x}}^1_{\Lambda}(\boldsymbol{z})\mathsf{P}\bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}}\right)+\mathsf{F}^0_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}}+\mathsf{F}^0_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})=0,$$

or

$$\mathsf{M}^0(\boldsymbol{z})\bar{\boldsymbol{x}}^0_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})=\mathsf{N}^0(\boldsymbol{z})\left[\begin{array}{ccccc} I & \bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}} & \bar{\boldsymbol{X}}^1_{\boldsymbol{\mathcal{E}}} & \cdots & \bar{\boldsymbol{X}}^N_{\boldsymbol{\mathcal{E}}}\end{array}\right]^{\mathsf{T}}.$$

We obtain the following expression for $\bar{\boldsymbol{X}}^j_{\boldsymbol{\mathcal{E}}}$ for $j=1,\ldots,N$

$$\bar{\boldsymbol{X}}^j_{\boldsymbol{\mathcal{E}}}=\left(\mathsf{D}^j\right)^{-1}\left(\int\left(\mathsf{R}^j_{\boldsymbol{z}}(\boldsymbol{z})+\mathsf{R}^j_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}^j_{\boldsymbol{z}}(\boldsymbol{z})\right)\mathsf{p}\bar{\boldsymbol{x}}^0_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega+\sum_{k=1}^{N-j}\mathsf{E}^j_k\bar{\boldsymbol{X}}^{j+k}_{\boldsymbol{\mathcal{E}}}\right)$$

which, when combined with

$$\int\mathsf{R}^0_{\boldsymbol{x}}(\boldsymbol{z})\bar{\boldsymbol{x}}^0_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})+\mathsf{R}^0_{\boldsymbol{X}}(\boldsymbol{z})\bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}}+\mathsf{R}^0_{\boldsymbol{X}+}(\boldsymbol{z})\left(\rho_{\boldsymbol{\Theta}}\bar{\boldsymbol{X}}^1_{\boldsymbol{\Theta}}+\bar{\boldsymbol{X}}^1_{\boldsymbol{\mathcal{E}}}+\bar{\boldsymbol{X}}^1_{\Lambda}\mathsf{P}\bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}}\right)+\mathsf{R}^0_{\boldsymbol{\mathcal{E}}}(\boldsymbol{z})d\Omega=0,$$

yields the linear system

$$\mathsf{O}\cdot\left[\begin{array}{ccccc} I & \bar{\boldsymbol{X}}^0_{\boldsymbol{\mathcal{E}}} & \bar{\boldsymbol{X}}^1_{\boldsymbol{\mathcal{E}}} & \cdots & \bar{\boldsymbol{X}}^N_{\boldsymbol{\mathcal{E}}}\end{array}\right]^{\mathsf{T}}=\mathsf{P}.$$

C Simulation and Clustering

To simulate an optimal policy at each date with N agents, we discretize the distribution across agents with K grid points that we find each period using a k-means clustering algorithm. Let $\{\boldsymbol{z}_i\}_{i=1}^N$ represent the current distribution of agents. The k-means algorithm generates K points $\{\bar{\boldsymbol{z}}_k\}_{k=1}^K$ with each agent i assigned to a cluster $k(i)$ to minimize the squared error $\sum_i \|\boldsymbol{z}_i - \bar{\boldsymbol{z}}_{k(i)}\|^2$. We let Ω represent the distribution of N agents and $\bar{\Omega}$ represent our approximating distribution of clusters.[27] At each history, we apply our algorithm to approximate the optimal policies around $\bar{\Omega}$. Moreover, by allowing σ to also scale deviations of Ω from $\bar{\Omega}$, we can increase the accuracy of our approximate the policy rules as follows. Let $\tilde{\boldsymbol{x}}(\sigma\boldsymbol{\varepsilon}, \sigma\boldsymbol{\mathcal{E}}, \sigma(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) + \bar{\boldsymbol{\Theta}}, z, \bar{\Omega} + \sigma(\Omega - \bar{\Omega}), \sigma(\Lambda - \bar{\Lambda}(\Omega)) + \bar{\Lambda}; \sigma)$ and $\tilde{\boldsymbol{X}}(\sigma\boldsymbol{\mathcal{E}}, \sigma(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) + \bar{\boldsymbol{\Theta}}, \bar{\Omega} + \sigma(\Omega - \bar{\Omega}), \sigma(\Lambda - \bar{\Lambda}(\Omega)); \sigma)$ denote the policy rules with scaling parameters σ. We can them approximate these policies with a Taylor expansion with respect to σ. For example, a first-order Taylor expansion would be

$$\tilde{\boldsymbol{x}}(\boldsymbol{\varepsilon}, \boldsymbol{\Theta}, \boldsymbol{z}, \Omega, \Lambda) \approx \bar{\boldsymbol{x}} + \left(\bar{\boldsymbol{x}}_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon} + \bar{\boldsymbol{x}}_{\boldsymbol{\mathcal{E}}}\boldsymbol{\mathcal{E}} + \bar{\boldsymbol{x}}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) + \partial\bar{\boldsymbol{x}} \cdot (\Omega - \bar{\Omega}) + \bar{\boldsymbol{x}}_{\Lambda}(\Lambda - \bar{\Lambda}) + \bar{\boldsymbol{x}}_{\sigma}\right)\sigma$$

and

$$\tilde{\boldsymbol{X}}(\boldsymbol{\Theta}, \Omega, \Lambda) \approx \bar{\boldsymbol{X}} + \left(\bar{\boldsymbol{X}}_{\boldsymbol{\mathcal{E}}}\boldsymbol{\mathcal{E}} + \bar{\boldsymbol{X}}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) + \partial\bar{\boldsymbol{X}} \cdot (\Omega - \bar{\Omega}) + \bar{\boldsymbol{X}}_{\Lambda}(\Lambda - \bar{\Lambda}(\Omega)) + \bar{\boldsymbol{X}}_{\sigma}\right)\sigma.$$

Both $\partial\bar{\boldsymbol{x}} \cdot (\Omega - \bar{\Omega})$ and $\partial\bar{\boldsymbol{X}} \cdot (\Omega - \bar{\Omega})$ are easily computed from terms already computed during the expansion, so these extra corrections have no additional computational cost.[28] Note that when $K = N$ we exactly approximate around Ω, but for $K < N$ we can speed up the computations by a factor of $\frac{N}{K}$. We choose K so that increasing K does not change the impulse responses reported in section 5 which results in a K of 2000 to 3000 and $N = 100000$ depending on the experiment.

D Competitive Equilibrium with fixed government policies

We demonstrate the application of our method outlined in section 3.1 to a problem of finding a competitive equilibrium for fixed government policies. We have two goals in mind: First,

[27] Formally $d\Omega(\boldsymbol{z}) = \sum_i \frac{1}{N}\delta(\boldsymbol{z} - \boldsymbol{z}_i)$ while $d\bar{\Omega}(\boldsymbol{z}) = \sum_i \frac{1}{N}\delta(\boldsymbol{z} - \bar{\boldsymbol{z}}_{k(i)})$.

[28] Following the steps in section (B.5) it can readily be seen that $\partial\bar{\boldsymbol{X}} \cdot (\Omega - \bar{\Omega}) = \int \mathsf{A}(\boldsymbol{z})(\boldsymbol{z} - \bar{\boldsymbol{z}}(\boldsymbol{z}))d\bar{\Omega}(\boldsymbol{z}) = \sum_i \frac{1}{N}\mathsf{A}(\bar{\boldsymbol{z}}_{k(i)})(\boldsymbol{z}_i - \bar{\boldsymbol{z}}_{k(i)})$ and $\partial\bar{\boldsymbol{x}}(\bar{\boldsymbol{z}}_k) = \mathsf{C}(\bar{\boldsymbol{z}}_k)\partial\bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\sigma}$.

the problem with fixed policies is simpler and explicit expressions for terms that arise in many of the steps in section 3 as well as those in appendix B can be derived. Second, we use this economy as a test case to analyze the accuracy of the our approximations by comparing the second-order expansions to solutions obtained with global methods.

To make our exposition most transparent, we assume that government has no expenditures, sets taxes $\Upsilon_t = T_t = 0$ and implements inflation $\Pi_t = 0$ in all periods. We assume all agents have equal ownership of firms and that there is no permanent component of labor productivity, $\theta_{i,t} = 0$. Finally, we assume that the aggregate shock is i.i.d., which removes Θ as a state variable. These assumptions deliver a version of Huggett (1993) economy with natural borrowing limits extended to allow for endogenous labor supply and aggregate shocks.

Individual decisions are expressed recursively. The aggregate variables depend on the realized aggregate shock $\mathcal{E}$ and the beginning of the period distribution of assets Ω. Since we assumed that government has no revenues, it also cannot issue debt, and so the distribution Ω satisfies $\int b d\Omega = 0$. We denote the space of such distributions by $\mathcal{W}$. We use tildes to denote policy functions, and let $\tilde{\boldsymbol{X}} = \begin{bmatrix} \tilde{Q} & \tilde{W} & \tilde{D} \end{bmatrix}^T$ be a vector of aggregate policy functions capturing interest rates, wages and dividends. Individual policy functions depend both on aggregate state $(\mathcal{E}, \Omega)$ and on idiosyncratic state (ε, b) where ε is the realization of the idiosyncratic shock that affects individual with asset beginning of period assets b. Let $\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{b} & \tilde{c} & \tilde{n} \end{bmatrix}^T$ be the triplet of the individual policy functions. Finally, $\widetilde{\Omega}(\mathcal{E}, \Omega) : \mathbb{R} \times \mathcal{W} \to \mathcal{W}$ be the law of motion describing how the aggregate distribution of debt next period is affected by the aggregate shock in the current period.

Individual optimality conditions consist of the budget constraint (2) and the optimality conditions (13)-(14). In our recursive notation, these conditions read

$$\tilde{c}(\varepsilon, \mathcal{E}, b, \Omega) + \tilde{Q}(\mathcal{E}, \Omega)\, \tilde{b}(\varepsilon, \mathcal{E}, b, \Omega) = \tilde{W}(\mathcal{E}, \Omega) \exp(\mathcal{E} + \varepsilon)\, \tilde{n}(\varepsilon, \mathcal{E}, b, \Omega) \tag{97a}$$

$$+ b + \tilde{D}(\mathcal{E}, \Omega),$$

$$\beta \mathbb{E} \left\{ u_c \left[\tilde{c}\left(\cdot, \cdot, \tilde{b}(\mathcal{E}, b, \Omega), \widetilde{\Omega}(\mathcal{E}, \Omega)\right) \right] \middle| \mathcal{E}, \Omega \right\} = \tilde{Q}(\mathcal{E}, \Omega)\, u_c \left[\tilde{c}(\varepsilon, \mathcal{E}, b, \Omega) \right], \tag{97b}$$

$$\tilde{W}(\mathcal{E}, \Omega) \exp(\mathcal{E} + \varepsilon)\, u_c \left[\tilde{c}(\varepsilon, \mathcal{E}, b, \Omega) \right] = -u_n \left[\tilde{n}(\varepsilon, \mathcal{E}, b, \Omega) \right], \tag{97c}$$

for all $\varepsilon, \mathcal{E}, b, \Omega$. These constraints construct the function F as in (20) of section 3. The aggregate constraints after imposing firm's optimality (15) can be written as

$$\left[\int \exp(\mathcal{E}+\varepsilon)\tilde{n}\left(\varepsilon,\mathcal{E},b,\Omega\right)d\Pr\left(\varepsilon\right)d\Omega\right]^{\alpha} = \int \tilde{c}\left(\varepsilon,\mathcal{E},b,\Omega\right)d\Pr\left(\varepsilon\right)d\Omega, \tag{98a}$$

$$\frac{\epsilon-1}{\epsilon}\alpha\left[\int \exp(\mathcal{E}+\varepsilon)\tilde{n}\left(\varepsilon,\mathcal{E},b,\Omega\right)d\Pr\left(\varepsilon\right)d\Omega\right]^{\alpha-1} = \tilde{W}\left(\mathcal{E},\Omega\right), \tag{98b}$$

$$\left(1-\frac{\epsilon-1}{\epsilon}\alpha\right)\left[\int \exp(\mathcal{E}+\varepsilon)\tilde{n}\left(\varepsilon,\mathcal{E},b,\Omega\right)d\Pr\left(\varepsilon\right)d\Omega\right]^{\alpha} = \tilde{D}\left(\mathcal{E},\Omega\right), \tag{98c}$$

for all $\mathcal{E},\Omega$ which construct the function (21) of section 3. Finally, the law of motion for the distribution of debts induced by the savings behavior of the agents is given by

$$\widetilde{\Omega}\left(\mathcal{E},\Omega\right)(y) = \int \iota\left(\tilde{b}\left(\varepsilon,\mathcal{E},b,\Omega\right)\leq y\right)d\Pr\left(\varepsilon\right)d\Omega \quad \forall y, \tag{99}$$

where ι is the indicator variable.

Equations (97), (98), and (99) fully describe the equilibrium behavior.

D.1 Points of expansion and zeroth-order terms

Consider the expansion around deterministic economy with a given distribution of assets Ω. Observe that we have

$$\bar{b}\left(b,\Omega\right) = b \text{ for all b,}\Omega. \tag{100}$$

This counterpart of lemma (1) implies that equations for deterministic economy are

$$\bar{c}\left(b,\Omega\right)+\bar{Q}\left(\Omega\right)b = \bar{W}\left(\Omega\right)\bar{n}\left(b,\Omega\right)+b+\bar{D}\left(\Omega\right), \tag{101a}$$

$$\beta = \bar{Q}\left(\Omega\right), \tag{101b}$$

$$\bar{W}\left(\Omega\right)u_c\left[\bar{c}\left(b,\Omega\right)\right] = -u_n\left[\bar{n}\left(b,\Omega\right)\right], \tag{101c}$$

for all (b,Ω) and aggregate constraints

$$\begin{aligned}\left[\int \bar{n}\left(b,\Omega\right)d\Omega\right]^{\alpha} &= \int \bar{c}\left(b,\Omega\right)d\Omega,\\ \frac{\epsilon-1}{\epsilon}\alpha\left[\int \bar{n}\left(b,\Omega\right)d\Omega\right]^{\alpha-1} &= \bar{W}\left(\Omega\right),\\ \left(1-\frac{\epsilon-1}{\epsilon}\alpha\right)\left[\int \bar{n}\left(b,\Omega\right)d\Omega\right]^{\alpha} &= \bar{D}\left(\Omega\right),\end{aligned} \tag{102}$$

and the law of motion

$$\bar{\Omega}\left(\Omega\right) = \Omega \tag{103}$$

that hold for all Ω.For a given Ω, we solve the system of equations above for $\bar{\boldsymbol{x}}(b,\Omega), \bar{\boldsymbol{X}}(\Omega)$.

To obtain the coefficients of the first-order expansion of the policy functions, we will need to know how policy functions are affected by perturbations to the individual state b. Differentiate (101a) and (101c) with respect to b to get a linear system and then solving for $\bar{\boldsymbol{x}}_b$ yields

$$\bar{\boldsymbol{x}}_b(b) \equiv \begin{bmatrix} \bar{b}_b(b) \\ \bar{c}_b(b) \\ \bar{n}_b(b) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \bar{Q}-1 & 1 & \bar{W} \\ 0 & \bar{W}u_{cc}\left[\bar{c}\left(b\right)\right] & u_{nn}\left[\bar{n}\left(b\right)\right] \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \bar{D} \\ 0 \end{bmatrix}, \tag{104}$$

where all the terms on the right hand side are known from the zeroth-order expansion.

D.2 First-order terms and factorization theorem

Consider the first order expansion with respect to σ of equations (98) and (97) and use method of undetermined coefficients to find the derivatives that multiply $\sigma\varepsilon$ and $\sigma\mathcal{E}$. The derivatives $\bar{\boldsymbol{x}}_\varepsilon(b)$ are easy to find since they cancel out from the expansions of the feasibility constraints and appear only in the individual optimality conditions

$$\bar{c}_\varepsilon\left(b\right) + \bar{Q}\bar{b}_\varepsilon\left(b\right) = \bar{W}\left(\bar{n}\left(b\right) + \bar{n}_\varepsilon\left(b\right)\right), \tag{105a}$$

$$\bar{Q}u_{cc}[\bar{c}(b)]\bar{c}_\varepsilon(b) = \beta u_{cc}[\bar{c}(b)]\bar{c}_b(b)\bar{b}_\varepsilon(b), \tag{105b}$$

$$\bar{W}\left(u_c\left[\bar{c}\left(b\right)\right] + u_{cc}\left[\bar{c}\left(b\right)\right]\bar{c}_\varepsilon\left(b\right)\right) = -u_{nn}\left[\bar{n}\left(b\right)\right]\bar{n}_\varepsilon\left(b\right). \tag{105c}$$

for all b. All variables apart from $\bar{b}_\varepsilon\left(b\right), \bar{c}_\varepsilon\left(b\right), \bar{n}_\varepsilon\left(b\right)$, are known from the zeroth-order expansion. Thus we can find $\bar{b}_\varepsilon\left(b\right), \bar{c}_\varepsilon\left(b\right), \bar{n}_\varepsilon\left(b\right)$ separately for each b by solving this 3×3 system of equations. In the direct analogy with (104) we can write this solution compactly as

$$\bar{\boldsymbol{x}}_\varepsilon(b) = \mathsf{E}(b)^{-1}\mathsf{G}(b) \tag{106}$$

for matrices $\mathsf{K}(b)$,$\mathsf{L}(b)$ known from the zeroth-order expansion. Similarly it can be shown that $\bar{\boldsymbol{X}}_\sigma, \bar{\boldsymbol{x}}_\sigma$ are all zero vectors.

Solving for the effect of the aggregate shocks is more complicated because they affect the distribution of debts and the aggregate constraints. Differentiating individual constraints implies that

$$\bar{c}_{\mathcal{E}}\left(b\right) + \bar{Q}\bar{b}_{\mathcal{E}}\left(b\right) + \bar{Q}_{\mathcal{E}}\bar{b}\left(b\right) = \bar{W}\left(\bar{n}\left(b\right) + \bar{n}_{\mathcal{E}}\left(b\right)\right) + \bar{W}_{\mathcal{E}}\bar{n}\left(b\right) + \bar{D}_{\mathcal{E}}, \tag{107a}$$

$$\beta u_{cc}[\bar{c}(b)]\left(\bar{c}_b(b)\bar{b}_{\mathcal{E}}(b) + \partial\bar{c}(b)\cdot\bar{\Omega}_{\mathcal{E}}\right) = \bar{Q}u_{cc}[\bar{c}(b)]\bar{c}_{\mathcal{E}}(b) + u_c[\bar{c}(b)]\bar{Q}_{\mathcal{E}}, \tag{107b}$$

$$\bar{W}\left(u_c\left[\bar{c}\left(b\right)\right] + u_{cc}\left[\bar{c}\left(b\right)\right]\bar{c}_{\mathcal{E}}\left(b\right)\right) + \bar{W}_{\mathcal{E}}u_c\left[\bar{c}\left(b\right)\right] = -u_{nn}\left[\bar{n}\left(b\right)\right]\bar{n}_{\mathcal{E}}\left(b\right). \tag{107c}$$

while differentiating the aggregate constraints gives

$$
\begin{aligned}
\alpha \bar{N}^{\alpha-1} \int \left(\bar{n}\left(b\right) + \bar{n}_{\mathcal{E}}\left(b\right)\right) d\Omega &= \int \bar{c}_{\mathcal{E}}\left(b\right) d\Omega, \\
\frac{\epsilon-1}{\epsilon}\alpha\left(\alpha-1\right)\bar{N}^{\alpha-2} \int \left(\bar{n}\left(b\right) + \bar{n}_{\mathcal{E}}\left(b\right)\right) d\Omega &= \bar{W}_{\mathcal{E}}, \\
\left(1 - \frac{\epsilon-1}{\epsilon}\alpha\right)\alpha \bar{N}^{\alpha-1} \int \left(\bar{n}\left(b\right) + \bar{n}_{\mathcal{E}}\left(b\right)\right) d\Omega &= \bar{D}_{\mathcal{E}},
\end{aligned}
\tag{108}
$$

where $\bar{N}$ is the aggregate labor supply in the zeroth-order expansion. Furthermore, by differentiating equation (99) with respect to $\mathcal{E}$ and evaluating at $\sigma = 0$ it can readily be seen that[29]

$$
\bar{\Omega}_{\mathcal{E}}(b) = -\omega(b)\bar{b}_{\mathcal{E}}\left(b\right) \text{ for all } b. \tag{109}
$$

This leads to the main difficulty of this problem: in order to solve for the response of aggregates $\bar{\boldsymbol{X}}_{\mathcal{E}}$ we must jointly solve equations (107a)-(108) for $\bar{\boldsymbol{x}}_{\mathcal{E}}(b)$ for all b. Our resolution is to apply the factorization theorem, theorem (1), to show that $\bar{\boldsymbol{x}}_{\mathcal{E}}(b)$ can be constructed as loadings on $\bar{\boldsymbol{X}}_{\mathcal{E}}$ and $\bar{\boldsymbol{X}}'_{\mathcal{E}} = \partial\bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\mathcal{E}}(b)$ which can be solved for independently.

Begin by differentiating the individual constraints with respect to Ω. Exploiting $\partial\bar{\Omega} = \mathbf{1}$ yields

$$
\begin{aligned}
\partial\bar{c}(b) + b\partial\bar{Q} + \beta\partial\bar{b}(b) &= \bar{n}(b)\partial\bar{W} + \bar{W}\partial\bar{n}(b) + \partial\bar{D} \\
\beta u_{cc}[\bar{c}(b)]\left(\bar{c}_b(b)\partial\bar{b}(b) + \partial\bar{c}(b)\right) &= u_c[\bar{c}(b)]\partial\bar{Q} + \beta u_{cc}[\bar{c}(b)]\partial\bar{c}(b) \\
\bar{W}u_{cc}[\bar{c}(b)]\partial\bar{c}(b) + u_c[\bar{c}(b)]\partial\bar{W} &= u_{nn}[\bar{n}(b)]\partial\bar{n}(b).
\end{aligned}
$$

This can be represented by the matrix expression

$$
\begin{bmatrix} \beta & 1 & -\bar{W} \\ \beta u_{cc}[\bar{c}(b)]\bar{c}_b(b) & 0 & 0 \\ 0 & \bar{W}u_{cc}[\bar{c}(b)] & -u_{nn}[\bar{n}(b)] \end{bmatrix} \begin{bmatrix} \partial\bar{b}(b) \\ \partial\bar{c}(b) \\ \partial\bar{n}(b) \end{bmatrix} = \begin{bmatrix} -b & \bar{n}(b) & 1 \\ u_c[\bar{c}(b)] & 0 & 0 \\ 0 & -u_c\left[\bar{c}\left(b\right)\right] & 0 \end{bmatrix} \begin{bmatrix} \partial\bar{Q} \\ \partial\bar{W} \\ \partial\bar{D} \end{bmatrix}
$$

which we can solve to yield the counterpart to the first result of theorem (1):

$$
\partial\bar{\boldsymbol{x}}(b) = \mathsf{C}(b)\partial\bar{\boldsymbol{X}}.
$$

To obtain the second result we differentiate equations (98a)-(98c) in an arbitrary direction

[29] For the derivation in detail see section B.5.

Δ evaluated at $\sigma = 0$ to find

$$\alpha \bar{N}^{\alpha-1} \left(\int \bar{n}(b)\, d\Omega \right) \cdot \Delta = \left(\int \bar{c}(b)\, d\Omega \right) \cdot \Delta$$
$$\frac{\epsilon - 1}{\epsilon} \alpha (\alpha - 1) \bar{N}^{\alpha-2} \left(\int \bar{n}(b)\, d\Omega \right) \cdot \Delta = \partial \bar{W} \cdot \Delta, \tag{110}$$
$$\left(1 - \frac{\epsilon - 1}{\epsilon} \alpha \right) \alpha \bar{N}^{\alpha-1} \left(\int \bar{n}(b)\, d\Omega \right) \cdot \Delta = \partial \bar{D} \cdot \Delta.$$

As noted in section (3),

$$\left(\int \bar{c}(b)\, d\Omega \right) \cdot \Delta = \int \partial \bar{c}(b) \cdot \Delta d\Omega + \int \bar{c}(b) d\Delta.$$

Evaluating this integral at $\Delta(b) = \bar{\Omega}_{\mathcal{E}}(b) = -\omega(b)\bar{b}_{\mathcal{E}}(b)$ and exploiting integration by parts we find

$$\left(\int \bar{c}(b)\, d\Omega \right) \cdot \bar{\Omega}_{\mathcal{E}} = \int \partial \bar{c}(b) \cdot \bar{\Omega}_{\mathcal{E}} d\Omega + \int \bar{c}(b) d\bar{\Omega}_{\mathcal{E}} = \int \mathsf{C}_1(b) d\Omega \partial \bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\mathcal{E}} + \int \bar{c}_b(b) \bar{b}_{\mathcal{E}}(b) d\Omega$$

where $\mathsf{C}_1(b)$ represents the first row of $\mathsf{C}(b)$. As a similar relationship can be derived for $\left(\int \bar{n}(b)\, d\Omega \right) \cdot \bar{\Omega}_{\mathcal{E}}$, plugging these into equations (110) yields a system of equations of the form

$$\mathsf{D} \partial \bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\mathcal{E}} = \int \mathsf{B}(b) \bar{\boldsymbol{x}}_{\mathcal{E}}(b) d\Omega$$

where D and $\mathsf{B}(b)$ are both known from zeroth-order terms. Solving for $\partial \bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\mathcal{E}}$ gives $\partial \bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\mathcal{E}} = \int \mathsf{A}(b) \bar{\boldsymbol{x}}_{\mathcal{E}}(b) d\Omega$. When combined with our expression for $\partial \bar{\boldsymbol{x}}(b)$, we get the analogue of (75) of theorem 1

$$\partial \bar{\boldsymbol{x}}(b) \cdot \bar{\Omega}_{\mathcal{E}} = \mathsf{C}(b) \int \mathsf{A}(b) \bar{\boldsymbol{x}}_{\mathcal{E}}(b) d\Omega.$$

Finally we can exploit this knowledge to solve for $\bar{\boldsymbol{X}}_{\mathcal{E}}$ and $\bar{\boldsymbol{x}}_{\mathcal{E}}(b)$ by first defining

$$\partial \bar{\boldsymbol{x}}(b) \cdot \bar{\Omega}_{\mathcal{E}} = \mathsf{C}(b) \partial \bar{\boldsymbol{X}} \cdot \bar{\Omega}_{\mathcal{E}} \equiv \mathsf{C}(b) \bar{\boldsymbol{X}}'_{\mathcal{E}}.$$

After substituting for $\partial \bar{\boldsymbol{x}}(b) \cdot \bar{\Omega}_{\mathcal{E}}$, equation (107) now defines a linear relationship between policy functions $\bar{\boldsymbol{x}}_{\mathcal{E}}(b)$ and aggregate variables $\begin{bmatrix} \bar{\boldsymbol{X}}_{\mathcal{E}} & \bar{\boldsymbol{X}}'_{\mathcal{E}} \end{bmatrix}$ which is independent of any other $\bar{\boldsymbol{x}}_{\mathcal{E}}(\hat{b})$. Thus, we can write (107) for each b as

$$\mathsf{M}(b) \bar{\boldsymbol{x}}_{\mathcal{E}}(b) = \mathsf{N}(b) \cdot \begin{bmatrix} \bar{\boldsymbol{X}}_{\mathcal{E}} & \bar{\boldsymbol{X}}'_{\mathcal{E}} \end{bmatrix}^{\mathsf{T}}, \tag{111}$$

where $\mathsf{M}(b)$ is a 3×3 matrix, and all coefficients of $\mathsf{M}(b)$ and $\mathsf{N}(b)$ are known from the zeroth-order expansion. Thus we can solve for each $\bar{\boldsymbol{x}}_{\mathcal{E}}(b)$ as a function of aggregate variables independently for each b. This simplifies analysis by breaking one system of $3K \times 3K$ equations into K system of 3×3 equations for each of K levels of debt. We can substitute the obtained expressions for $\bar{\boldsymbol{x}}_{\mathcal{E}}(\cdot)$ into equations (108) and

$$\bar{\boldsymbol{X}}'_{\mathcal{E}} = \int \mathsf{A}(b)\bar{\boldsymbol{x}}_{\mathcal{E}}(b)d\Omega \tag{112}$$

to generate a 6×6 system of equations of the form

$$\mathsf{O} \cdot \begin{bmatrix} \bar{\boldsymbol{X}}_{\mathcal{E}} & \bar{\boldsymbol{X}}'_{\mathcal{E}} \end{bmatrix}^{\mathsf{T}} = \mathsf{P}. \tag{113}$$

Equation (113) can be easily solved numerically for $\bar{\boldsymbol{X}}_{\mathcal{E}}, \bar{\boldsymbol{X}}'_{\mathcal{E}}$. This completes the solution for the first-order responses to the aggregate shock $\mathcal{E}$.

D.3 Second and higher order terms

To construct a second-order expansion of policy functions we need additional terms. These are $\bar{\boldsymbol{x}}_{\varepsilon\varepsilon}(b), \bar{\boldsymbol{x}}_{\varepsilon\sigma}(b), \bar{\boldsymbol{x}}_{\mathcal{E}\mathcal{E}}(b), \bar{\boldsymbol{x}}_{\mathcal{E}\sigma}(b)$, $\bar{\boldsymbol{x}}_{\sigma\sigma}(b), \bar{\boldsymbol{X}}_{\mathcal{E}\mathcal{E}}, \bar{\boldsymbol{X}}_{\mathcal{E}\sigma}, \bar{\boldsymbol{X}}_{\sigma\sigma}$. Calculations of $\bar{\boldsymbol{x}}_{\varepsilon\varepsilon}$,$\bar{\boldsymbol{x}}_{\mathcal{E}\mathcal{E}}$,$\bar{\boldsymbol{X}}_{\mathcal{E}\mathcal{E}}$ then proceeds analogously to their first-order counterparts, while the cross-partials $\bar{\boldsymbol{x}}_{\varepsilon\sigma}, \bar{\boldsymbol{x}}_{\mathcal{E}\sigma}, \bar{\boldsymbol{X}}_{\mathcal{E}\sigma}$ are all zeros. Unlike the first-order expansion, the intercept terms $\bar{\boldsymbol{x}}_{\sigma\sigma}, \bar{\boldsymbol{X}}_{\sigma\sigma}$ are no longer zero. They depend on $\text{var}(\varepsilon), \text{var}(\mathcal{E})$ and capture such effects as precautionary savings. Solution for $\bar{\boldsymbol{x}}_{\sigma\sigma}(b), \bar{\boldsymbol{X}}_{\sigma\sigma}$ involves steps similar to those used to solve for $\bar{\boldsymbol{x}}_{\mathcal{E}}(b), \bar{\boldsymbol{X}}_{\mathcal{E}}$ in the previous section. Details are provided in section B.6.

D.4 Numerical implementation and accuracy

To implement our algorithm numerically, we approximate Ω with a discrete distributions with K points $\{b_k\}_k$ with masses $\{\omega_k\}_k$. All the integrals in our expressions collapse then to sums. For example, our expression (112) for $\boldsymbol{X}'_{\mathcal{E}}$ becomes

$$\bar{\boldsymbol{X}}'_{\mathcal{E}} = \sum_k \mathsf{A}(b_k)\omega_k \bar{b}_{\mathcal{E}}(b_k).$$

All intermediate terms, such as $\mathsf{E}(b_k)$, can be computed independently for each k, making the algorithm highly parallelizable. Once we compute approximations of the policy functions in the current period, we use Monte-Carlo methods described in section C to obtain the next period distribution of assets $\tilde{\Omega}$, for which we repeat this procedure.

We now discuss the numerical accuracy of our approximations. To compare our (second-order) approximated policy rules with those solved via global methods we shut down the aggregate shocks and compute the steady state distribution of assets the model presented in section D.3. Optimal policies of each agent are computed using the endogenous grid method of Carol (2005). The steady-state distribution is approximated using a histogram and computed using the transition matrix constructed from the policy rules following Young (2010). We then compare the approximated policy rules using our method, around the same stationary distribution, to those of the global solution. As noted in section D.3, even in absence of aggregate shocks our techniques are still necessary, for a second-order expansion, to determine the effect of the presence idiosyncratic risk on policies through $\bar{\boldsymbol{x}}_{\sigma\sigma}$ and $\bar{\boldsymbol{X}}_{\sigma\sigma}$.

We calibrate the 6 parameters of this model, ν,γ,β, α, ϵ and σ_ε, as follows. We set $\nu = 1$ and $\gamma = 2$ to match of calibration in section 4. β is set to target an interest rate of 2%. $\epsilon = 6$ targets a markup of 20% and the decreasing returns to scale parameter, α, is chosen to target a labor share of 0.66. The standard deviation σ_ε is set at .25 to match our calibration in section 4. We choose very loose ad-hoc borrowing constraints, $\underline{b} = -100$ to approximate a natural debt limit. To ensure an accurate approximation to the global policy rules, we approximated the consumption and labor policy rules using cubic splines with 200 grid points and the steady-state distribution with a histogram with 1000 bins.

Agents' policies and aggregate variables are then approximated using a second-order approximation around the stationary distribution $\bar{\Omega}$ with the expressions derived in this section. The equilibrium policy rules and aggregates, around this distribution of 1000 points, can be approximated using single processor in 0.8 seconds. The equilibrium interest rate found by perturbation code was 2.0013%. The percent error in the aggregate labor supply was 0.036%. Finally, we can evaluate the policy errors for the individual agents. In the first panel of figure VIII we plot the percentage errors for consumption relative to the global solution for a median shock to ς along with a ± 2.5 standard deviation shock.

For almost all of the agents, the perturbation methods perform well with errors less than 0.1% which would correspond making an error in consumption of less than a dollar on every $1000 dollars spent. As expected, since the quadratic approximation assumes natural borrowing limits, the accuracy deteriorates near the borrowing limit assumed by the global solution. The range plotted in figure VIII contains 99.2% of the agents in the stationary distribution. As a robustness check, we also observe how accuracy behaves as we change σ_ε. For values of $\sigma_\varepsilon \in [0.1, 0.5]$, we compute the average absolute percentage error as

$$\int\int \frac{\left|c^{global}(\sigma_\varepsilon\varepsilon, b; \sigma_\varepsilon) - c^{pertub}(\sigma_\varepsilon\varepsilon, b; \sigma_\varepsilon)\right|}{c^{global}(\sigma_\varepsilon\varepsilon, b; \sigma_\varepsilon)} d\Pr(\varepsilon) d\Omega$$

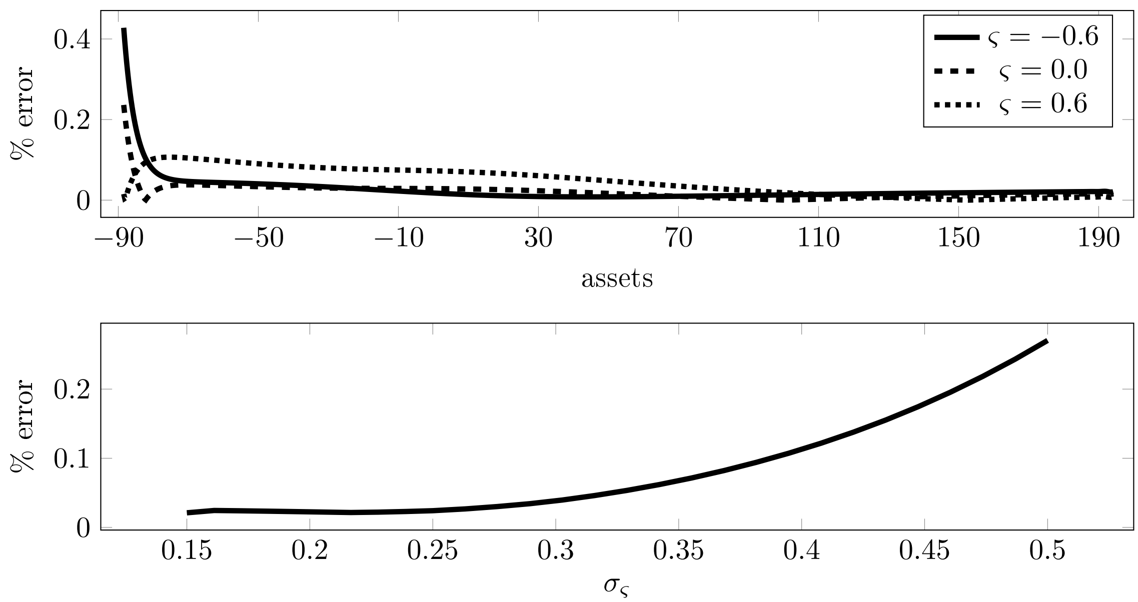

Figure VIII: Percentage error of consumption policy functions relative to global solution in top panel. Bottom panel plots average (with respect to the stationary distribution) absolute consumption error relative to global solution as σ_ς varies.

and plot it in the bottom panel of figure VIII. We see that the average error remains very low around the level of our calibration and increases moderately for higher σ_ε.

E Robustness

We provide details for several cases that are discussed in sections 5 and 6 but omitted from the main text. These include stationary TFP shocks, alternative welfare criteria, no inequality shocks $f = 0$, and no menu costs $\psi = 0$.

Stationary TFP shocks are modeled as an AR(1) process (see equation 9a) in the text, with autocorrelation $\rho_\Theta = 0.95^4$ and std of $\mathcal{E}_\Theta = 3\%$. All other parameters and the initial condition are kept as the same as the baseline calibration in section 4. The impulse responses are computed to a one standard deviation negative shock. As mentioned in section 5.2, we extend the definition of "natural rate" for our heterogeneous agent economy to be the real interest rates in a competitive equilibrium with $\psi = 0$ that implements flexible prices, fiscal policy set Υ_t at the non-stochastic optimum and initial conditions as in section 4. The optimal response of the nominal rate in Figure IX is plotted as a deviation from this natural rate and we see that it quite closely tracks the response in the baseline with the i.i.d growth rate specification.

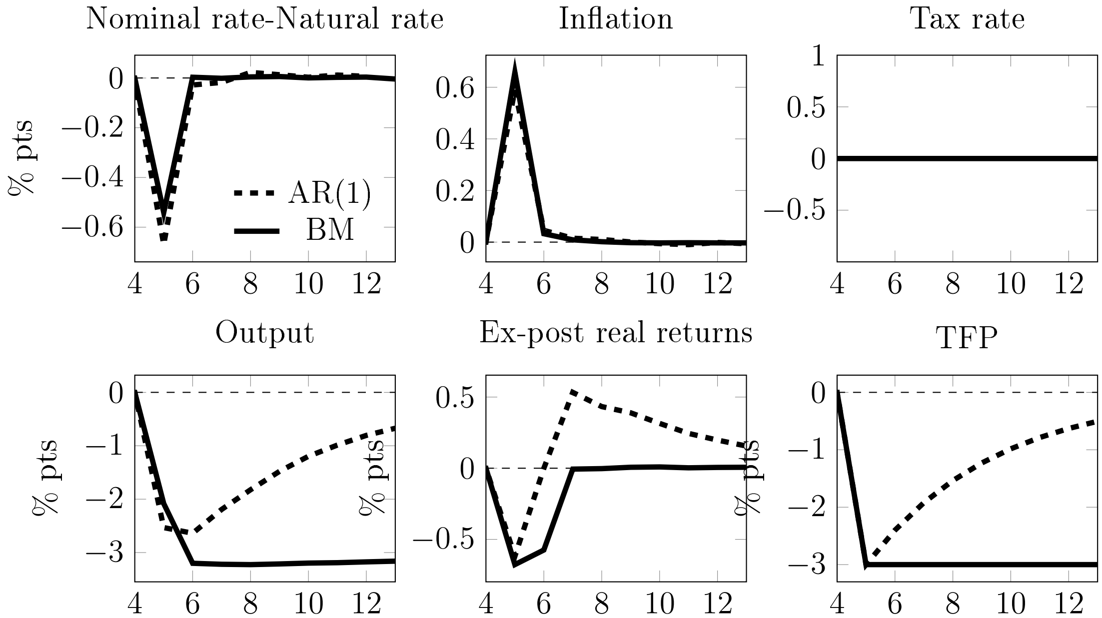

Figure IX: Optimal monetary response to an AR(1) productivity shock. The solid line is the baseline calibration with i.i.d growth rate shocks.

To illustrate the role of alternative preferences for the planner, we modify the objective function (12) to introduce time-invariant Pareto weights $\omega_i \propto \theta_{-1,i}^{\alpha_\omega}$. We calibrate the α_ω such that the optimal non-stochastic tax rate equals 24%. Our estimate suggests an $\alpha_\omega = 0.75$ indicating higher Pareto weights to productive agents which rationalizes low marginal tax rates. We keep all other parameters and the initial condition as in the baseline calibration and compute optimal monetary responses to a TFP shock in figure X and the markup shock in figure XI. The fact that the responses are similar to the utilitarian case in the baseline underlines that changes in Pareto wights primarily map into changes in the steady state level of the tax rate while the response of policies to shocks is governed by insurance motives arising mainly due to incomplete markets.

Next we turn to the case where we study TFP shocks with $f = 0$, implying that the aggregate TFP shock results in the same proportional change labor productivity for all agents. As mentioned in the text, even in absence of changes in inequality, a permanent drop in wages with agents heterogeneous in asset holdings creates needs for insurance. The exercise in figure XII allows us decompose the effect into the part that comes because initial wealth differences and the part that comes because of heterogeneous exogenous exposure to the shock.

Finally we study the case without nominal rigidities by setting $\psi \approx 0$. In response to a TFP shock (figure XIII), the planner provides insurance through inflation which generates

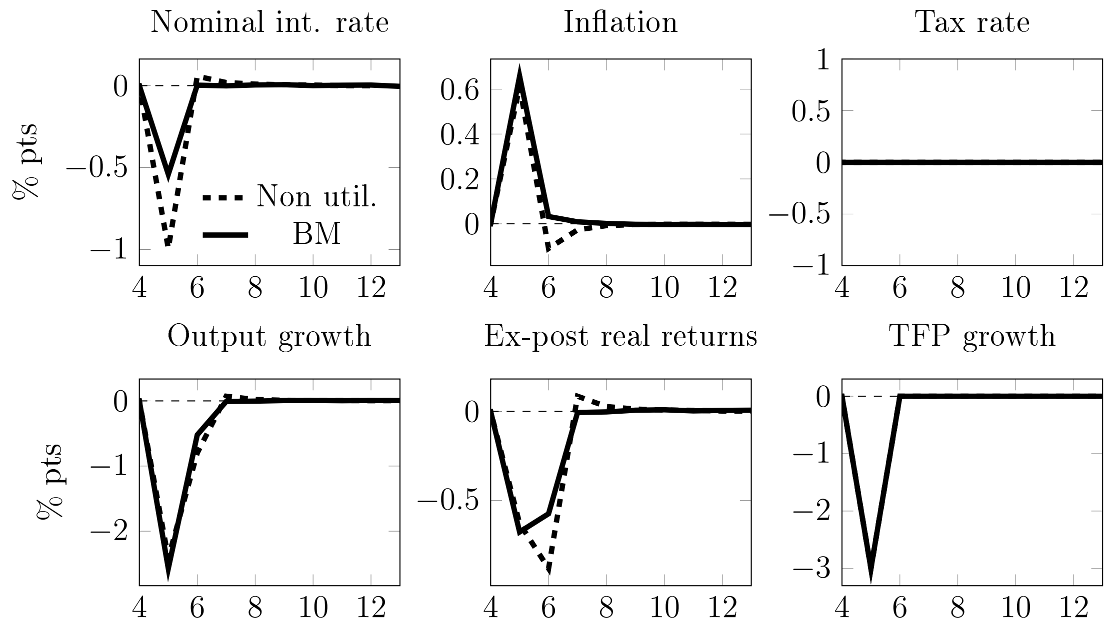

Figure X: Optimal monetary response to a productivity shock with non-utilitarian Pareto weights. The solid line is the baseline calibration with utilitarian Pareto weights

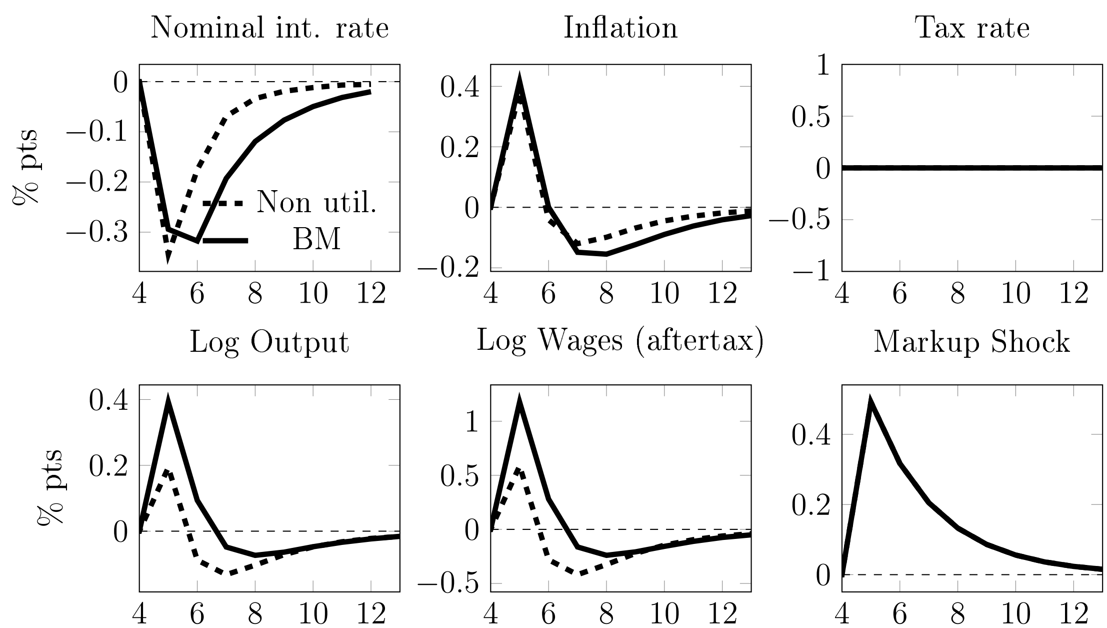

Figure XI: Optimal monetary response to a markup shock with non utilitarian Pareto weights. The solid line is the baseline calibration with utilitarian Pareto weights

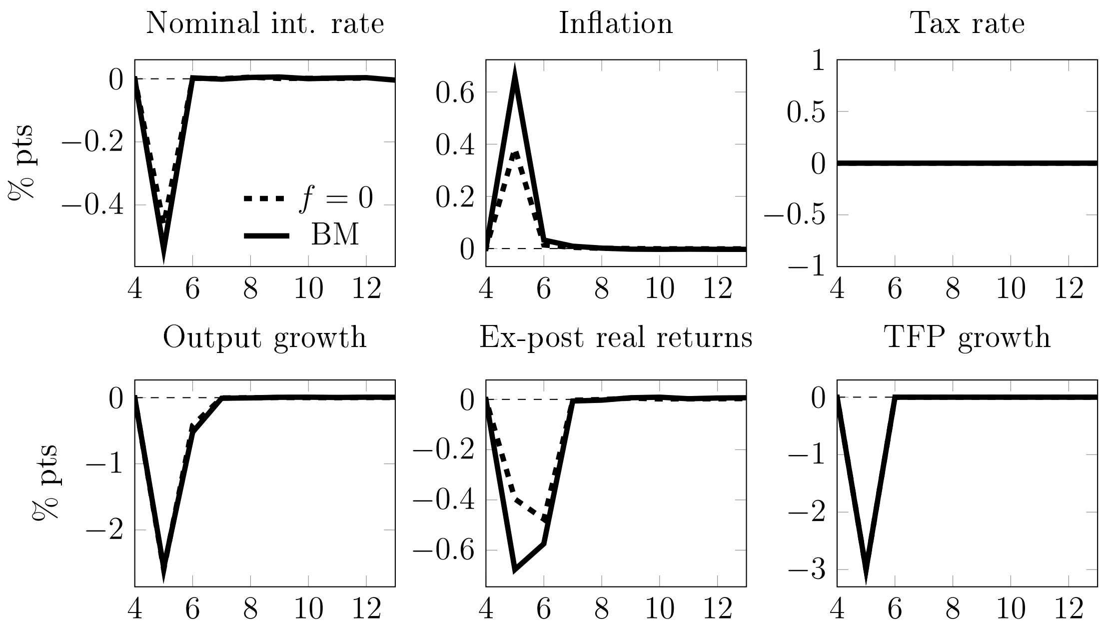

Figure XII: Optimal monetary response to a productivity shock with $f = 0$. The solid line is the baseline case with f calibrated as in section 4.

state contingent returns. The size of response is about 3 to 4 times larger than the baseline mainly driven by the fact that price changes are less costly and an effective way of insuring against permanent TFP shocks. In response to a markup shock (figure XIV), we see that real wages drop and then follow the path of the shock. This is quite different from the baseline where planner increased real wages to redistribute to towards the wage earners from equity holders. The reason for this difference is that absent nominal rigidities, the ability of the planner to affect aggregate demand and wages through changes in nominal rates is substantially diminished. The planner still induces some inflation on impact because bonds and equities are correlated and lowering returns partially redistributes towards low wage agents.

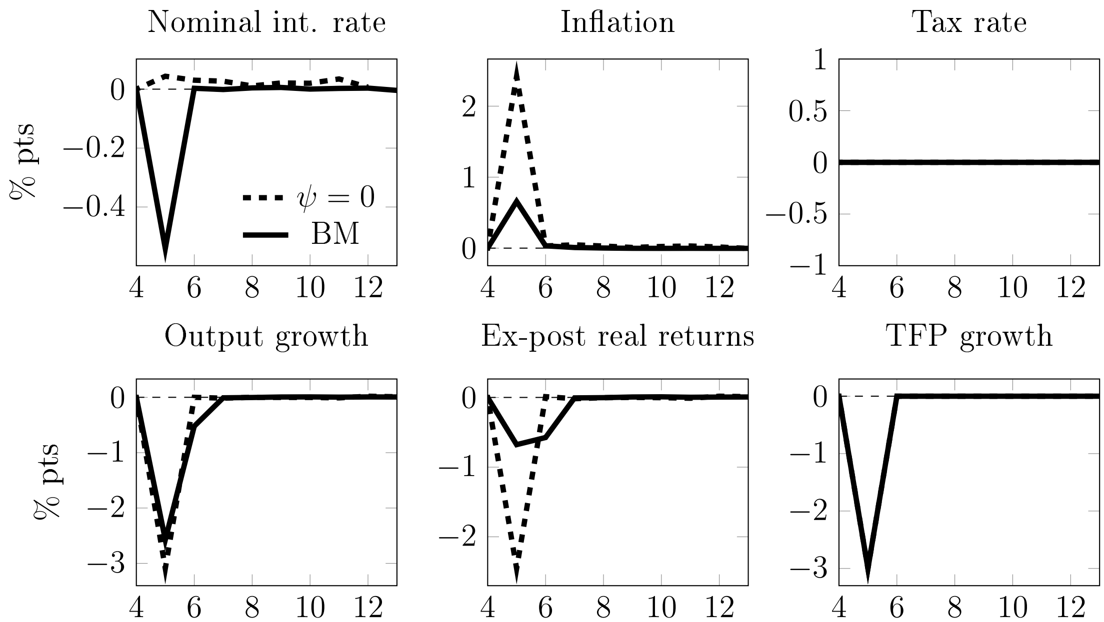

Figure XIII: Optimal monetary response to a productivity shock with $\psi = 0$. The solid line is the baseline calibration with $\psi = 20$.

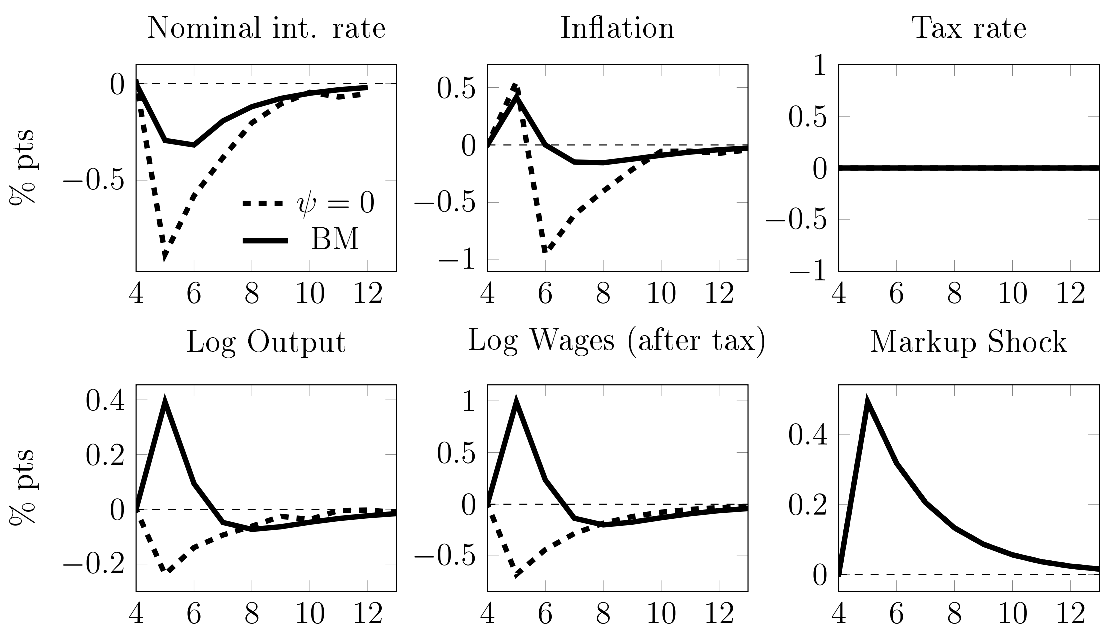

Figure XIV: Optimal monetary response to a markup shock with $\psi = 0$. The solid line is the baseline calibration with $\psi = 20$.